Stiffness and Flexibility Methods for Structural Analysis

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Abstract

This document presents the classical methods of flexibility and stiffness for the analysis of statically indeterminate structures

1 Preliminary Definitions

Statically Determinate Structure: A structure such that when subjected to any set of external forces, moments and(or) pressures; its reactions and internal stress fields can be computed using only equations of equilibrium.

Degree of Freedom: The number of scalars necessary to describe the deformed shape of a structure.

Degree of Indeterminancy: Set of internal forces/reactions which if known, would make the structure statically determinate. This set is not unique, but their number is.

2 Flexibility Method

*Definition:*The flexibility coefficient *i,j* is the deformation/rotation at point i due to a unit load/moment at point j. The flexibility coefficient *i,j* is denoted as $f_{i,j}$.

The flexibility method for the analysis of statically indeterminate structures operates using the principles of superposition and proportionality. In the flexibility method, a statically indeterminate structure with degree of indeterminancy equal to n is analyzed as a statically determinate "released" structure with n releases (selected by the analyst). These releases correspond to external boundary conditions (support reactions) or continuity constraints, such as a hinge in a point of structural continuity. The forces/moments corresponding to each release are treated as unknown and are found by formulating compatibility equations in order to restore continuity or a specified boundary condition at the point of release. The basic equation of the flexibility method is

$$\mathbf{F}\mathbf{f} + \mathbf{x}_o = \mathbf{x}_p \tag{1}$$

where $\mathbf{F}$ is the flexibility matrix of the released structure, $\mathbf{x}$ is the hyperstatic force vector (forces/moments corresponding to each release), $\mathbf{x}_o$ is the deformation vector at the release points induced by the loads on the released structure, and $\mathbf{x}_p$ is the permanent deformation vector of the statically indeterminate structure at the released points.

The solution of the hyperstatic forces is given by

$$\mathbf{f} = \mathbf{F}^{-1}(\mathbf{x}_p - \mathbf{x}_o) \tag{2}$$

2.1 Computing Flexibility Matrices in 3-D Frame Structures

The flexibility coefficient *i,j* for a 3D frame structure is given by

$$f_{i,j} = \int_L \frac{n_j \bar{n}_i}{AE} dx + \int_L \frac{v_{2,j} \bar{v}_{2,i}}{Ga_r} dx + \int_L \frac{v_{3,j} \bar{v}_{3,i}}{Ga_r} dx + \int_L \frac{t_j \bar{t}_i}{GJ} dx + \int_L \frac{m_{2,j} \bar{m}_{2,i}}{EI_2} dx + \int_L \frac{m_{3,j} \bar{m}_{3,i}}{EI_3} dx \tag{3}$$

where lower case variables have been used to indicate that the internal forces are produced by unit loads and the subscripts i or j indicate whether the internal forces are generated by unit loads at i or j respectively. As can be seen interchanging the indeces i and j does not change the value of the flexibility coefficient, therefore we can conclude that $f_{i,j} = f_{j,i}$ and thus the flexibility matrix is a symmetric matrix.

3 Stiffness Method

Definition: The stiffness coefficient *i,j* is the force/moment at degree of freedom i necessary to generate a unit displacement/rotation at degree of freedom j while all other degrees of freedom are restrained to zero displacement/rotation.

The basic equation for the stiffness method is

$$\mathbf{K}\mathbf{x} = \mathbf{f}_e \tag{4}$$

where $\mathbf{K}$ is the stiffness matrix of the structure, $\mathbf{x}$ is the displacement at all degrees of freedom and $\mathbf{f}_e$ is the vector of external forces/moments applied at the degrees of freedom. The solution for the displacement at all degrees of freedom is given by

$$\mathbf{x} = \mathbf{K}^{-1}\mathbf{f}_e \tag{5}$$

3.1 Computing Stiffness Matrices

The stiffness coefficient *i,j* of a structure can be computed using the virtual work theorem by evaluating the following integral throughout the volume of the structure

$$k_{i,j} = \int_V \sigma_i^T \bar{\epsilon}_j dV = \int_V \epsilon_i^T \mathbf{E} \bar{\epsilon}_j dV \tag{6}$$

where $\bar{\epsilon}_j$ is the virtual strain field corresponding to the unit virtual displacement at DOF j, ϵ_i is the strain field corresponding to a unit displacement at DOF i

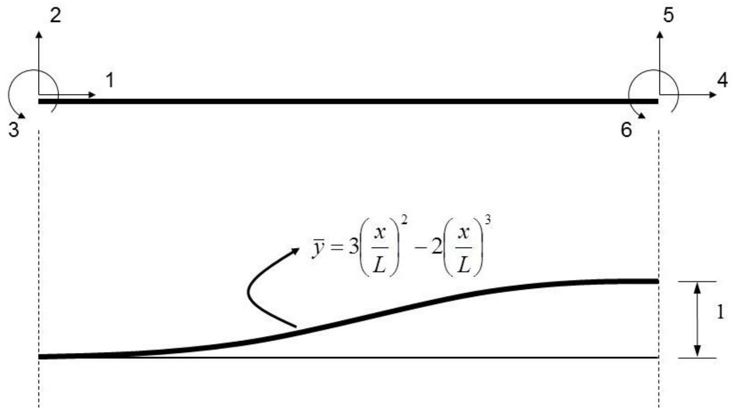

Figure 1: Example of application of virtual work theorem to compute stiffness coefficient in prismatic member

and **E** is the elasticity matrix.
To illustrate consider the prismatic frame element shown in Fig.1. Suppose we are interested in computing the stiffness coefficient $k_{5,5}$. In this case one has

$$\epsilon = \bar{\epsilon} = \frac{6}{L^2} - \frac{12}{L^3}x \tag{7}$$

which results in

$$k_{5,5} = \int_0^L \left(\frac{6}{L^2} - \frac{12}{L^3}x\right)^2 EIdx = \frac{12EI}{L^3} \tag{8}$$

Other coefficients can be computed similarly.

4 Relationship between Flexibility and Stiffness

In a linear elastic structure, the relationship between the forces and displacement at various points can be synthesized by the following linear matrix equation

$$\mathbf{x} = \mathbf{F}\mathbf{f} \tag{9}$$

where $\mathbf{x}$ is the displacement vector, $\mathbf{f}$ is the force vector and $\mathbf{F}$ is the flexibility matrix whose components are defined by eq.3. Conversely, the relationship between the forces and displacement can also be synthesized as follows

$$\mathbf{f} = \mathbf{K}\mathbf{x} \tag{10}$$

where $\mathbf{x}$ is the displacement vector, $\mathbf{f}$ is the force vector and $\mathbf{K}$ is the stiffness matrix whose components are defined by eq.10. Upon comparison of the previous two equations it is evident that given a finite set of DOF, the stiffness matrix $\mathbf{K}$ and the flexibility matrix $\mathbf{F}$ are related as follows

$$\mathbf{F} = \mathbf{K}^{-1} \tag{11}$$