Matrix Analysis - A Brief Review

Eric M. Hernandez,Ph.D.

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Abstract

This document is intended to provide a brief and concise review of some essential aspects of matrix analysis, specially as it concerns its application for structural analysis.

1 Vectors and Vector Spaces - Intuitive Concepts

A vector is a mathematical representation of a point in n-dimensional space, where n is a positive integer. We shall use bold lower case letters such as $\mathbf{a}, \mathbf{q}, \mathbf{z}$ to denote vectors. Lower case cursive letters such as a, q, z will be used to represent scalars. We shall denote the i^{th} component of a vector $\mathbf{v}$ as v_i. Vectors are typically written as a column of numbers. As an example consider the vector $\mathbf{x}$ representing a point in 3-dimensional space

$$\mathbf{x} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \qquad x_2 = -3 \tag{1}$$

However a vector need not be exclusively used to represent points in a physical space. We can also use vectors to represent "points" in more general and abstract spaces. For example consider generating a vector to represent a particular color. By using the additive primary colors RED-BLUE-GREEN it is possible to obtain many other colors. Therefore one can think of the primary colors as three orthogonal axes and any particular color as a "point" in a three-dimensional space. By defining the three primary colors as

$$red = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad blue = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad green = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{2}$$

we obtain for example that

$$yellow = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad cyan = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad magenta = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{3}$$

Purely mathematical examples of vector spaces can also be constructed. Consider all possible monic quartic polynomials of the form

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + x^4 = 0 \tag{4}$$

It is possible to define the vector space of all quartic polynomials as a 4-dimensional space where each coefficient is an independent "axis". So that the vector

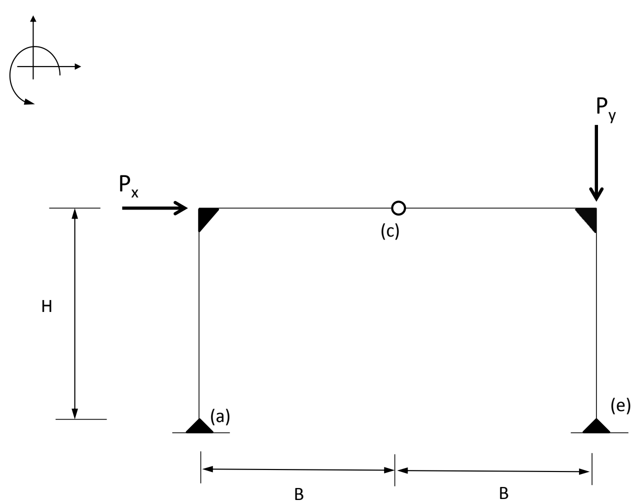
$$\mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ 3 \\ 5 \end{bmatrix} \tag{5}$$

represents the polynomial

$$-1 + 2x + 3x^2 + 5x^3 + x^4 = 0 \tag{6}$$

which itself is a "point" is a four-dimensional space. Another way to define a vector space of polynomials is by their roots, In this case you would have n roots for a polynomial of degree n. The roots taken as a vector represent a point in n dimensional space.

Vector spaces with a more structural flavor can also be constructed. Consider the three-hinge frame shown below



This structure has four independent reaction forces; two at point (a), A_x and A_y, and two at point (e), E_x and E_y. We can think of these reactions as a point in a four dimensional space.

$$\mathbf{r} = \begin{bmatrix} A_x \\ A_y \\ E_x \\ E_y \end{bmatrix} \tag{7}$$

2 Matrices

In its simplest form a matrix is a rectangular arrangement of numbers, however a matrix can also be a rectangular arrangement of functions, and even other matrices! We will represent matrices by a capital bold letter of the English alphabet. We shall denote the component in the i^{th} row and the j^{th} column of a matrix $\mathbf{A}$ as $a_{i,j}$, such that if

$$\mathbf{A} = \begin{bmatrix} 0 & 4 & -1 \\ 9 & 6 & -3 \\ 1 & 2 & 5 \end{bmatrix} \qquad a_{3,1} = 1 \tag{8}$$

A matrix $\mathbf{A}$ defined over the field of real numbers with n rows and m columns will be denoted as $\mathbf{A} \in \mathbb{R}^{n \times m}$. Matrices constitute an indispensable tool in modern engineering, applied science and mathematics. Among a multitude of potential applications, matrices are a convenient and compact way to represent, analyze and solve linear set of equations, such as the ones typically found in the analysis of equilibrium and deformation of linear elastic structures. Consider as a simple example the equilibrium equations of the frame shown previously

$$\Sigma F_x = A_x + E_x = -P_x \tag{9}$$

$$\Sigma F_y = A_y + E_y = P_y \tag{10}$$

$$\Sigma M_{c,left} = -A_y B + A_x H = 0 \tag{11}$$

$$\Sigma M_{c,right} = E_y B + E_x H = P_y B \tag{12}$$

which can be conveniently and compactly expressed in matrix form as

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ H & -B & 0 & 0 \\ 0 & 0 & H & B \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ E_x \\ E_y \end{bmatrix} = \begin{bmatrix} -P_x \\ P_y \\ 0 \\ P_y B \end{bmatrix} \tag{13}$$

The solution to the previous equation can be easily found using well-known matrix analysis methods. Matrices can be studied from the perspective of linear operator theory, i.e., a matrix $\mathbf{A}$ operates on a vector $\mathbf{v}$ and it transforms it to a vector $\mathbf{w}$. This is the most general and complete approach to matrix analysis. From this perspective, matrix analysis is mainly concerned with: (i)providing a qualitative understanding of matrices as operators on linear vector spaces and (ii)developing numerical methods to quantify their properties and operations. We will make extensive use of MATLAB to perform the matrix computations required in Advanced Structural Analysis. To this end we provide along with the theory, references to essential MATLAB functions for numerical matrix analysis.

In essence, our objective in this course will be to represent the behavior of a linear elastic structure by a matrix and forces and deformations as vectors. *Thus, by understanding how matrices operate on vectors, we shall understand how structures respond to the action of forces and deformations.*

3 Matrix Analysis using MATLAB

In the MATLAB environment a vector such as

$$\mathbf{x} = \begin{bmatrix} 1 \\ 7 \\ -3 \end{bmatrix} \tag{14}$$

can be specified by typing the following commands

$$\texttt{v=zeros(3,1)} \tag{15}$$
$$\texttt{v(1,1)=1} \tag{16}$$
$$\texttt{v(2,1)=7} \tag{17}$$
$$\texttt{v(1,1)=-3} \tag{18}$$

Alternatively, it can also be specified as

$$\texttt{v=[1;7;-3]} \tag{19}$$

In the case of the matrix in eq.8, it can be specified as

$$\texttt{A=[0 4 -1;9 6 -3;1 2 5]} \tag{20}$$

and the command to extract, say the element in the third row and first column is

$$\texttt{A(3,1)} = 1 \tag{21}$$

4 Vectors and Vector Spaces - Formal Definitions

In this section we provide more formal mathematical definitions of vector spaces and its application to matrix analysis.

4.1 Vector Spaces

Vector Space: A vector space is a set V of objects (called vectors in a general sense, but not necessarily referring only to physical quantities such as position,

velocity, force, etc.) closed under addition, which is associative, commutative and possesses an identity element and additive inverses in the set. The set is also closed under scalar multiplication. Vector spaces can be finite dimensional or infinite dimensional. In structural applications we are interested mostly in finite dimensional vector spaces defined over the field of real numbers.
Vector Subspace: A subspace U of a vector space V is a subset of V and it is a vector space itself.
Linear dependence and independence: A set of vectors $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$ in a vector space is said to be *linearly dependent* if there are scalars $a_1, a_2, ..., a_n$ such that

$$a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + ... + a_n\mathbf{x}_n = 0 \tag{22}$$

A subset of V that is not linearly dependent is said to be *linearly independent*.

Basis: A subset S of a vector space V is said to span V if every element of V may be represented as a linear combination of elements of S. A linearly independent set which spans a vector space V is called a basis for V and denoted as $\mathcal{B}^V$ or simply $\mathcal{B}$ if the vector space is implicit. This means every element of V can be represented in terms of the basis vectors in one and only one way. However, a basis is not unique. A member of an n^{th} dimensional basis $\mathcal{B}$ is denoted as $\mathcal{B}_i$ for $i = 1, 2, ..., n$. Typically when representing basis in matrix form, each column of the matrix represents one basis vector.

Dimension: For finite dimensional vector spaces, the number of elements in the basis of V is its dimension and it is denoted as $dimV$. All basis of V have the same dimension.

4.1.1 Example

Consider the position of a object in three dimensional space (denoted as $\mathbf{R}^3$). This is a vector space of dimension 3 and can be spanned by the following basis

$$\mathcal{B}_1 = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 4 \\ 1 & 1 & 3 \end{bmatrix} \tag{23}$$

but also by the following

$$\mathcal{B}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 3 \end{bmatrix} \tag{24}$$

and clearly by the most usual

$$\mathcal{B}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{25}$$

5 Basic Operations

This section describes the basic operations that will be typically performed in our course using matrices and vectors. Note that in the context of this exposition, a vector is simply a spacial case of a matrix.

5.1 Addition

The elements of the matrix $\mathbf{C} = \mathbf{A} + \mathbf{B}$ are defined as

$$c_{i,j} = a_{i,j} + b_{i,j} \tag{26}$$

This clearly indicates that only matrices of the same size can be added. In MATLAB, the addition of matrices is simply implemented by using the command `+`, and thus after specifying `A` and `B`, the matrix `C` can be found by typing

$$\texttt{C=A+B} \tag{27}$$

Vector addition is simply a special case of matrix addition, where only the first subindex remains (since vectors are single column matrices). Therefore the components of $\mathbf{z} = \mathbf{x} + \mathbf{y}$ are given by

$$z_i = x_i + y_i \tag{28}$$

5.2 Multiplication

The components of the matrix product $\mathbf{Z} = \mathbf{QR}$ are defined as

$$z_{i,j} = \sum_k q_{i,k} r_{k,j} \tag{29}$$

Thus two matrices $\mathbf{Q} \in \mathbb{R}^{n \times m}$ and $\mathbf{R} \in \mathbb{R}^{p \times r}$ can only be multiplied in the standard sense if $m = p$. The resulting matrix $\mathbf{Z}$ will be of dimension $n \times r$.

Four special cases result from this general formulation:
-*multiplication of two vectors - inner product*: The inner product of two vectors results in a scalar and it is defined as

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y} = \mathbf{y}^T \mathbf{x} = \sum_i x_i y_i \tag{30}$$

Two vectors are said to be orthogonal if their inner product is zero.

-*multiplication of two vectors - outer product*: The outer product of two vectors results in a matrix $\mathbf{M}$ and it is defined as

$$m_{j,k} = xy^T = x_j y_k \tag{31}$$

- *pre-multiplication of a matrix times a vector*: Pre-multiplying a matrix $\mathbf{A}$ times a vector $\mathbf{v}$ results in a vector $\mathbf{y}$ which can be defined as

$$\mathbf{y} = \mathbf{A}\mathbf{v} \tag{32}$$

where

$$y_i = \sum_j a_{i,j} v_j \tag{33}$$

or put in words, it is the linear combination of the columns of $\mathbf{A}$ weighted by the components of the vector $\mathbf{v}$.

- *pre-multiplication of a matrix times a scalar*: Pre-multiplying a matrix $\mathbf{A}$ times a scalar α results in a matrix $\mathbf{B}$ which can be defined as

$$\mathbf{B} = \mathbf{A}\alpha \tag{34}$$

where

$$b_{i,j} = a_{i,j}\alpha \tag{35}$$

In MATLAB, multiplication can be specified by using the command `*` between the elements (vectors or matrices) being multiplied, i.e. `A * B`.

6 Coordinates of a vector

A basis $\mathcal{B}$ for a vector space can be represented as a matrix, where every column represents a member of the basis. Therefore any vector $\mathbf{v}$ of the vector space V of dimension n can be represented as a linear combination of the basis vectors $\mathbf{b}_i$ such that

$$\mathbf{v} = \sum_i \mathbf{b}_i x_i = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{B}\mathbf{x} \tag{36}$$

where x_i is the coordinate of the vector $\mathbf{v}$ along the direction $\mathbf{b}_i$. A *standard basis* $\mathcal{B}_S^V$ of a vector space V is one in which the coordinates of a vector are the vector itself, thus for an n dimensional space, a standard basis is represented by the identity matrix

$$\mathcal{B}_S = \mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix} \tag{37}$$

7 Coordinate Transformation

As discussed previously, a basis $\mathcal{B}$ for a vector space V is not unique and one could transition from one basis to another. Let's denote $\mathbf{A}$ as the matrix of basis vectors for the basis $\mathcal{B}_1$ and $\mathbf{B}$ as the matrix of basis vectors for basis $\mathcal{B}_2$. Every column of $\mathbf{A}$ can be expressed in the basis $\mathcal{B}_2$ as

$$\mathbf{a}_i = \mathbf{B}\mathbf{t}_i \tag{38}$$

which can be written for all vector as

$$\mathbf{A} = \mathbf{B}\mathbf{T} \tag{39}$$

Therefore a vector $\mathbf{v}$ expressed in basis $\mathcal{B}_1$ as

$$\mathbf{v} = \mathbf{A}\mathbf{x} \tag{40}$$

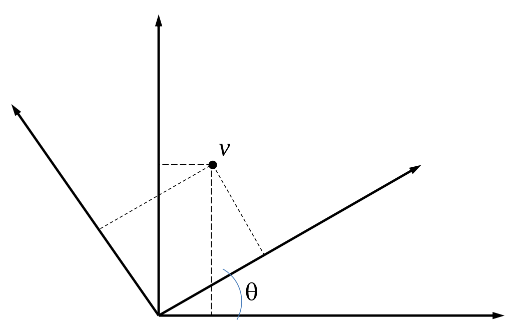
can be expressed in the basis $\mathcal{B}_2$ by substituting $\mathbf{A} = \mathbf{B}\mathbf{T}$ to get

$$\mathbf{v} = \mathbf{B}\mathbf{T}\mathbf{x} = \mathbf{B}\mathbf{y} \tag{41}$$

where $\mathbf{y} = \mathbf{T}\mathbf{x}$ are the coordinates of $\mathbf{x}$ in the basis $\mathcal{B}_2$ with basis vectors as columns of $\mathbf{B}$.

In the special case of a rotation θ the 2-D Cartesian frame of reference, the coordinate transformation matrix $\mathbf{T}$ is given by

$$T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{42}$$



The figure above is the graphical interpretation of the 2-D coordinate transformation matrix in eq.42.

8 Matrix Operator Coordinate Transformation

A matrix $\mathbf{Q}$ operating on a vector $\mathbf{z}$ expressed in basis $\mathcal{B}_1$ results in a vector $\mathbf{w}$

$$\mathbf{w} = \mathbf{Q}\mathbf{z} \tag{43}$$

transforming both vectors in eq.43 to a different basis $\mathcal{B}_2$ and substituting gives

$$\mathbf{z} = \mathbf{T}\mathbf{x} \tag{44}$$

$$\mathbf{w} = \mathbf{T}\mathbf{y} \tag{45}$$

$$\mathbf{T}\mathbf{y} = \mathbf{Q}\mathbf{T}\mathbf{x} \tag{46}$$

$$\mathbf{y} = \mathbf{T}^{-1}\mathbf{Q}\mathbf{T}\mathbf{x} \tag{47}$$

which means that the matrix operator $\mathbf{Q}$ originally given in basis $\mathcal{B}_1$ can be expressed in $\mathcal{B}_2$ as

$$\mathbf{T}^{-1}\mathbf{Q}\mathbf{T} \tag{48}$$

This relationship will become extremely useful when expressing equilibrium and displacement equations in different coordinate systems.

9 Transpose of a Matrix

The transpose of a matrix $\mathbf{A}$ is denoted as $\mathbf{B} = \mathbf{A}^T$ and it is defined as

$$b_{i,j} = a_{j,i} \tag{49}$$

A matrix $\mathbf{A}$ is called symmetric if

$$a_{i,j} = a_{j,i} \tag{50}$$

To compute the transpose of a matrix $\mathbf{A}$ in MATLAB, the command is `A'`

10 Rank of a Matrix

The rank of a matrix is defined as the maximum number of linearly independent column (or row) vectors. A matrix is called full column rank if its rank is equal to the number of columns. Similarly it is called full row rank, if its rank is equal to the number of rows. If a matrix is square and it is full column (or row) rank we simply refer to it as full rank. It can be shown that for any matrix the row rank is equal to the column rank. In MATLAB the command to compute the rank of a matrix $\mathbf{A}$ is `rank(A)`.

11 Inverse of a Matrix

The inverse matrix $\mathbf{B}$ of a matrix $\mathbf{A}$ is defined as

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I} \tag{51}$$

and it is denoted as $\mathbf{B} = \mathbf{A}^{-1}$. The inverse of a matrix, in the strict sense, only exists for square matrices with full rank. In MATLAB the inverse of matrix $\mathbf{A}$ is given by the command `inv(A)`. A linear set of equations given by

$$\mathbf{y} = \mathbf{Ax} \tag{52}$$

multiplying both sides by the inverse of $\mathbf{A}$, namely $\mathbf{A}^{-1}$

$$\mathbf{A}^{-1}\mathbf{y} = \mathbf{A}^{-1}\mathbf{Ax} \tag{53}$$

resulting in

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y} \tag{54}$$

This provides the solution for $\mathbf{x}$.

12 Matrix Partitions

A matrix can be partitioned in multiple ways, one useful partition in structural analysis is to partition a matrix into four blocks

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \tag{55}$$

where $\mathbf{A}_{ij}$ is a sub-matrix. As an example consider

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 6 & 7 \\ -3 & 4 & 9 & 1 \\ 1 & 3 & 3 & 0 \\ 0 & 2 & -8 & 10 \end{bmatrix} \tag{56}$$

one possible partition is

$$\mathbf{A}_{11} = \begin{bmatrix} 1 & 2 & 6 \\ -3 & 4 & 9 \\ 1 & 3 & 3 \end{bmatrix} \tag{57}$$

$$\mathbf{A}_{12} = \begin{bmatrix} 7 \\ 1 \\ 0 \end{bmatrix} \tag{58}$$

$$\mathbf{A}_{21} = \begin{bmatrix} 0 & 2 & -8 \end{bmatrix} \tag{59}$$

$$\mathbf{A}_{22} = \begin{bmatrix} 10 \end{bmatrix} \tag{60}$$

In MATLAB these partitions can be obtained with the following commands

```
A(1:3,1:3)   (61)
A(1:3,4)     (62)
A(4,1:3)     (63)
A(4,4)       (64)
```

It can be shown that the partitions of $\mathbf{A}^{-1}$ can be expressed as a function of the partitions of $\mathbf{A}$ as

$$\mathbf{A}^{-1} = \begin{bmatrix} (\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21})^{-1} & \mathbf{A}_{11}^{-1}\mathbf{A}_{12}(\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} - \mathbf{A}_{22})^{-1} \\ (\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} - \mathbf{A}_{22})^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & (\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12})^{-1} \end{bmatrix} \quad (65)$$

Any square submatrix of another matrix is called a principal submatrix.

13 Null Space

The null space of a matrix $\mathbf{A}$ is defined as the subspace containing all the vectors $\mathbf{v}$ which satisfy

$$\mathbf{A}\mathbf{v} = 0 \quad (66)$$

In MATLAB the null space of a matrix $\mathbf{A}$ is computed by using the command `null(A)`. It can be shown that full column rank matrices do not have a null space. The converse is also true, if a matrix has a null space then it can not be full column rank.

14 Determinant

The determinant of a matrix $\mathbf{A} \in \mathbb{R}^{n,n}$ is a scalar denoted as $det(\mathbf{A})$ and defined by

$$det(\mathbf{A}) = \sum_{\sigma} sgn(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)} \quad (67)$$

where the sum runs over all $n!$ permutations σ of the n element sequence $(1, 2, ..., n)$. The $sgn(.)$ function of a permutation is either $+1$ or -1 depending if the permutation is even or odd. A permutation is even if the minimum number of interchanges necessary to restore the permutation to its natural state of $(1, 2, ..., n)$ is even, and it is odd otherwise. For example, for $n = 3$, the permutation $(1, 3, 2)$ is odd, while the permutation $(3, 1, 2)$ is even. Can you find another criteria which easily defines if a permutation is even or odd?

The determinant of a matrix is an important quantity for many reasons, one important one is that the determinant of a rank deficient matrix is equal to zero. This will become very useful when defining the eigenvalues of a matrix.

In MATLAB the determinant of a matrix $\mathbf{A}$ is computed by using the command, `det(A)`.

15 Positive Definite Matrices

A matrix $\mathbf{A}$ is positive definite *iff*

$$\mathbf{x}^T\mathbf{A}\mathbf{x} > 0 \quad \forall x \tag{68}$$

Positive definite matrices are of great importance for physics and engineering applications since they represent stable systems with quadratic potential energy energy functions.

A matrix $\mathbf{A}$ is positive semi-definite iff

$$\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0 \quad \forall x \tag{69}$$

It can be shown (Rayleigh theorem) that the ratio

$$r = \frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \tag{70}$$

is maximized whenever $\mathbf{x} = \phi_1$ and minimized whenever $\mathbf{x} = \phi_n$.

Fundamental Theorem on Positive Semi-Definite Matrices
Let $\mathbf{A}$be a square symmetric matrix, then the following properties are equivalent:

- $\mathbf{A}$ is positive semi-definite
- $\mathbf{A} + \epsilon\mathbf{I}$ is positive definite $\forall\epsilon > 0$
- All the eigenvalues of all the principal submatrices of $\mathbf{A}$ are non-negative
- All eigenvalues of $\mathbf{A}$ are non-negative

Furthermore

- If $\mathbf{A}$ is positive definite it is invertible
- If $\mathbf{A}$ is positive definite its inverse $\mathbf{A}^{-1}$ is also positive definite

16 Norms

In its simplest form a norm is a measure of the length of a vector. The three most popular and useful norms for vectors are:

16.1 Euclidean Norm

The Euclidean norm or simply the 2-norm is an extension of Pythagoras theorem to n dimensions. The Euclidean norm of a vector $\mathbf{x}$ is represented as $||\mathbf{x}||_2$ and is defined as:

$$||\mathbf{x}||_2 = \left(\sum_{k=1}^{n} x_k^2 \right)^{1/2} \tag{71}$$

16.2 Manhattan Norm

The Manhattan norm or simply the 1-norm is represented as $||\mathbf{x}||_1$ and is defined as:

$$||\mathbf{x}||_1 = \sum_{k=1}^{n} |x_k| \tag{72}$$

16.3 Infinity Norm

The infinity norm is represented as $||\mathbf{x}||_\infty$ and is defined as:

$$||\mathbf{x}||_\infty = max\left(|x_k|\right) \tag{73}$$

In the context of MATLAB the command `norm(x,n)` provides the n-norm of a vector.

17 Eigenvalues and Eigenvectors

When a matrix $\mathbf{A}$ acts upon a vector $\mathbf{x}$ by pre-multiplication, the result is another vector $\mathbf{y}$, whose direction and magnitude are in general different from those of $\mathbf{x}$, however there are some vectors, which we shall refer to as eigenvectors ϕ, which possess the special property that

$$\mathbf{A}\phi = \lambda\phi \tag{74}$$

where λ is a scalar, which we define as the eigenvalue corresponding to the eigenvector ϕ. This indicates that eigenvectors are such that whenever acted upon by the matrix, they simply "stretch" or "contract" depending on the magnitude of the corresponding eigenvalue.

All $n \times n$ full rank symmetric matrices possess n linearly independent eigenvectors and n different corresponding eigenvalues. Furthermore, if the matrix is real and symmetric the eigenvalues will all be real or come in complex conjugate pairs (Can you prove that?).

Rearranging eq.74 we find that

$$(\mathbf{A} - \mathbf{I}\lambda_i)\,\phi_i = 0 \quad \forall i \tag{75}$$

which shows that the eigenvector ϕ_i corresponding to eigenvalue λ_i is in the null space of the matrix $(\mathbf{A} - \mathbf{I}\lambda_i)$. This leads to the fact that

$$det\,(\mathbf{A} - \mathbf{I}\lambda_i) = 0 \quad \forall i \tag{76}$$

From which all eigenvalues of $\mathbf{A}$ can be computed. The polynomial resulting from eq.76 is known as the characteristic polynomial. It can be shown (Caley-Hamilton theorem) that every matrix satisfies its own characteristic polynomial. In MATLAB the coefficients of the characteristic polynomial of a matrix $\mathbf{A}$ can be obtained by using the command `poly(A)`. In MATLAB, the eigenvalues and eigenvectors of a matrix $\mathbf{A}^{n\times n}$ are computed with the following command

```
[v,d]=eig(A)
```
(77)

where `v` is the matrix of eigenvectors arranged in columns and `d` is a diagonal matrix where every diagonal entry is an eigenvalue corresponding to the eigenvector in column coincident with the location of the diagonal entry of `d` such that

```
A*v(:,j)=d(j,j)*v(:,j)
```
(78)

for any scalar `j` between 1 and n. Finally, eigenvalues are assigned numbers as a function of their magnitude, that is, for an $n \times n$ matrix λ_1 is the eigenvalue with the largest magnitude and λ_n the eigenvalue with the smallest magnitude.