Virtual Work in Structural Analysis

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September 1, 2016

Abstract

This document presents the theorem of virtual work and some of its applications for structural analysis. It is shown that the most important aspects of structural analysis (computing forces and reactions) can be reduced to applications of the theorems of virtual work and complementary virtual work.

1 Theorem of Virtual Work - Virtual Displacements

The theorem of virtual work can be stated as:

Theorem 1. *The virtual work produced by a set of forces F_i moving through differential virtual displacements $\bar{\Delta}$ is equal to the internal virtual strain energy produced by the internal stresses σ (in equilibrium with F) acting on the differential virtual strains $\bar{\epsilon}$ (compatible with virtual displacements $\bar{\Delta}$).*

$$\sum_i F_i \bar{\Delta}_i = \int_V \sigma^T \bar{\epsilon} dV \tag{1}$$

where F_i are forces in equilibrium with the internal stress field σ and $\bar{\Delta}$ are virtual displacements compatible with the virtual internal strain field $\bar{\epsilon}$.

2 Theorem of Complementary Virtual Work - Virtual Forces

The theorem of complementary virtual work can be stated as:

Theorem 2. *The complementary virtual work produced by a set of virtual forces $\bar{F}_i$ moving through displacements Δ is equal to the internal virtual strain energy produced by the internal virtual stresses $\bar{\sigma}$ (in equilibrium with $\bar{F}$) acting on the strains ϵ (compatible with the displacements Δ).*

$$\sum_i \bar{F}_i \Delta_i = \int_V \bar{\sigma}^T \epsilon dV \tag{2}$$

where $\bar{F}_i$ are virtual forces in equilibrium with the virtual stress field $\bar{\sigma}$ and Δ are the real displacements of the structure compatible with the real internal strain field ϵ.

3 Computing Deformations using Virtual Work

The virtual work theorem as stated in eq.2 can be used to compute deformations in structures by simplifying the left side of the equality and using $\bar{F}_1 = 1$ resulting in

$$\bar{1}\Delta = \int_V \bar{\sigma}^T \epsilon dV = \int_V \bar{\sigma}^T \mathbf{E}^{-1} \sigma dV \tag{3}$$

where the virtual unit load $\bar{1}$ is applied at the point and in the direction of the desired deformation Δ. The virtual stress field $\bar{\sigma}$ is generated by the unit virtual load, σ is the real stress field produced by the real loads which induce the real displacement Δ and $\mathbf{E}$ is the elasticity tensor defined as the relationship between stress and strain $\sigma = \mathbf{E}\epsilon$ at every point in the structure.

3.1 Application to 3-D Framed Structures

Stresses in three dimensional framed members can be resolved into 6 independent internal resultant forces as shown in fig.1

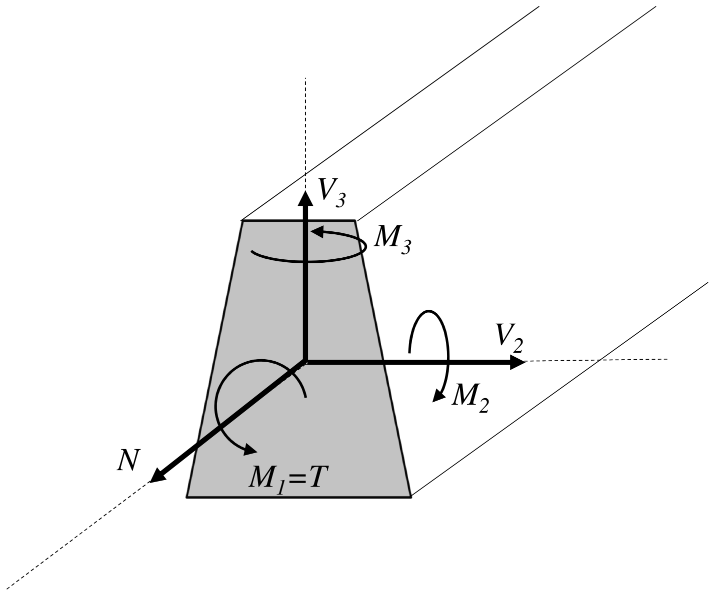


Figure 1: Internal forces in frames

Consequently eq.3 can be computed based on the internal resultant member forces as

$$\Delta = \int_L \frac{N\bar{n}}{AE}dx + \int_L \frac{V_2\bar{v}_2}{Ga_r}dx + \int_L \frac{V_3\bar{v}_3}{Ga_r}dx + \int_L \frac{T\bar{t}}{GJ}dx + \int_L \frac{M_2\bar{m}_2}{EI_2}dx + \int_L \frac{M_3\bar{m}_3}{EI_3}dx \tag{4}$$

where L is the length of the structure, A is the cross sectional area, a_r is the effective shear area, I_2 and I_3 are the cross sectional moment of inertia about local axes 2 and 3 respectively, J is the cross section torsional constant, E is Young's modulus and G is the shear modulus of the material. The above equation is only valid for linear elastic members. If the member is prismatic and homogenous the elastic and section properties become constants, otherwise they become functions of space.

4 Computing Forces using Virtual Work

To compute reactions and internal forces using virtual work, one must substitute the orginal structure by one in which the reaction or internal force of interest is treated as an unknown external force and proceed to apply eq.1.

4.1 Statically Determinate Structures

For statically determinate structures it is evident that the virtual displacement field will occur without any internal strains, since removing any external or internal restraint results in a mechanism, therefore eq.1 simplyfies to

$$\sum_i F_i \bar{\Delta}_i = 0 \tag{5}$$

From this equation one unknown will result which can be computed directly. Also note that since the right-hand side of the equation is zero, only relative values of the virtual displacement field are important.

In general the procedure is simple and the steps can be summarized as follows: 1) Define which force(reaction or internal) is to be determined, 2) Remove the restrain corresponding to that force, 3) determine the shape of the resulting mechanism upon a virtual displacement and finally 4) compute the virtual work of all the forces including the unkown forces, in this step is essential to account for the sign of the work,that is, if the virtual displacement and the force are in opposite directions then the work is negative. This procedure will result in one equation and one unknown, for which a solution follows immediatly.

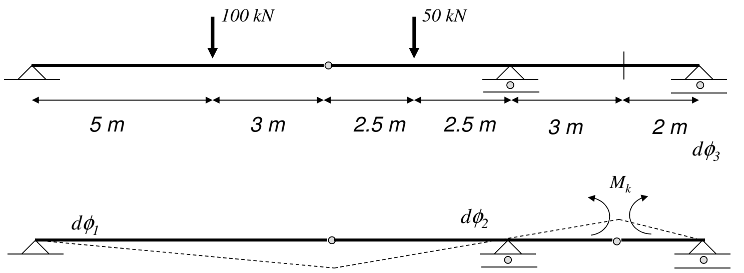
Fig.2 shows example where the internal bending moment in a Gerber beam is computed using virtual work (see Fig.2).

4.2 Statically Indeterminate Structures

Internal or external forces in statically indeterminate structures can be computed using the virtual work postulated in eq.1

$$F_i \bar{1}_i = \int_V \sigma^T \bar{\epsilon} dV \tag{6}$$

where σ represents the stress field corresponding to the real loads, F_i is the force of interest at point "i", $\bar{1}_i$ is a compatible unit virtual displacement at point "i", σ is the stress field corresponding to the real loads and $\bar{\epsilon}$ is the virtual



$$d\phi_2 = \frac{8}{5} d\phi_1 \qquad d\phi_3 = \frac{3}{2} d\phi_2$$

$$d\phi_3 = \frac{12}{5} d\phi_1$$

$$\sum dW = 100(5\,d\phi_1) + 50(2.5\,d\phi_2) + M_k\,d\phi_2 + M_k\,d\phi_3 = 0$$

$$\sum dW = 100(5\,d\phi_1) + 50(\frac{5}{2}\frac{8}{5} d\phi_1) + M_k \frac{8}{5} d\phi_1 + M_k \frac{12}{5} d\phi_1 = 0 \longrightarrow M_k = -175\ kNm$$

Figure 2: Application of virtual work theorem to determine internal bending moment

strain field compatible with the unit virtual displacement $\bar{1}_i$ and the boundary conditions.

To illustrate consider computing the central reaction in a 2-span primatic continuous beam subject to a uniform distributed load shown in fig.3

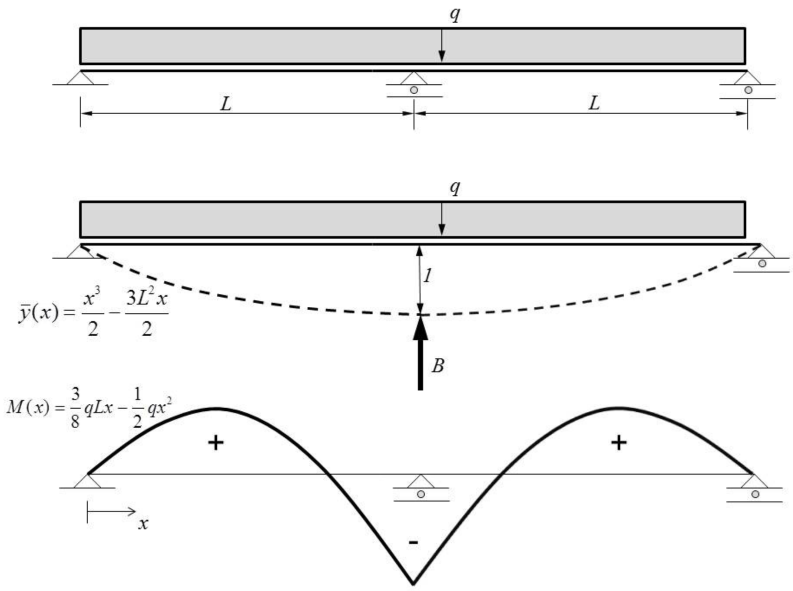


Figure 3: Example of application of virtual work theorem to compute reaction in statically indeterminate structure

$$\sigma = \frac{M(x)z}{I} \tag{7}$$

$$\bar{\epsilon} = z\frac{d^2\bar{y}(x)}{dx^2} = z\frac{3}{L^2}\frac{x}{L} \tag{8}$$

Substituting the definitions of σ and $\bar{\epsilon}$ into eq.6 we find

$$2\int_0^L q\left(\frac{1}{2}\left(\frac{x}{L}\right)^3 - \frac{3}{2}\left(\frac{x}{L}\right)\right)dx + B = \frac{2}{L^3}\int_0^L \left(\frac{3}{8}qLx - \frac{1}{2}qx^2\right)(3x)\,dx \tag{9}$$

$$2\int_0^L q\bar{y}dx + B = 0 \tag{10}$$

$$B = \frac{10}{8}qL \tag{11}$$

5 Müller-Breslau Principle

The Müller-Breslau principle states that the shape of the influence line of a structure (statically determinate or indeterminate) coincides with the shape of a compatible unit virtual displacement at the restrain corresponding to the force of interest(reaction or internal). This shape then needs to be calibrated to obtain the true influence line. This calibration is done by simply placing a unit load in a location where the the result is known, the values at all other locations can be found by elementary geometrical relationships.

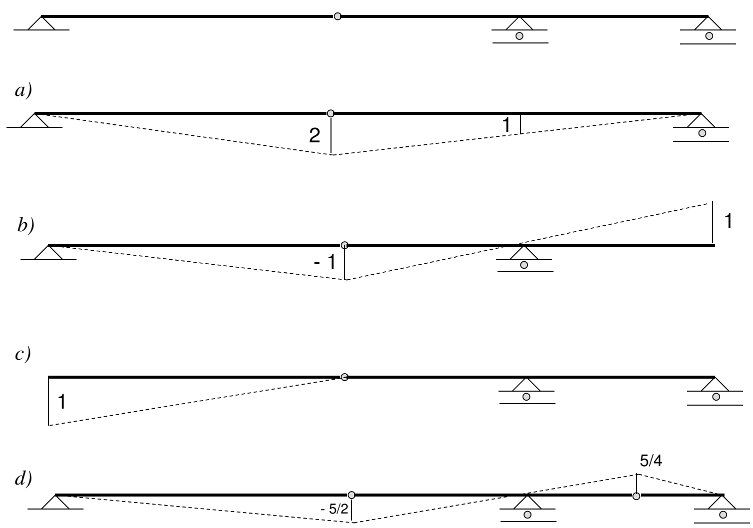


Figure 4: Influence Lines for different forces

Examples of how to calibrate the influence lines is shown in Fig.5 (geometry from fig.3 and 4). In case *a*) the unit load is placed on top of the middle support, and the reaction of 1 is obvious. In case *b*) the unit load is placed on top of the right support to again obtain a value of 1 at that location. In case *c*) the load is placed in the left support to obtain the calibrating value of 1 and finally in case *d*) the load is place in the center of the right span to obtain a bending moment of 5/4.