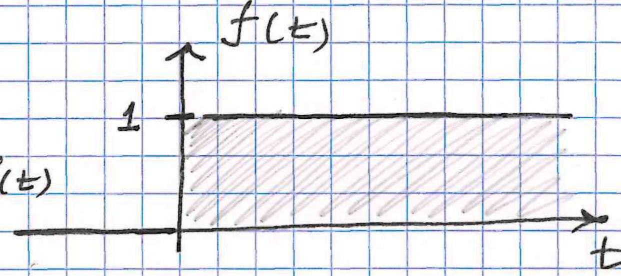
Determine response of SDOF to step function



$$m\ddot{x} + c\dot{x} + kx = f(t)$$

$$\ddot{x} + 2\omega_n \xi \dot{x} + \omega_n^2 x = \frac{1}{m} f(t)$$

$$\omega_n^2 = \frac{k}{m}$$

$$c = 2m\omega_n \xi$$

STEP 1: Take Laplace TRANSFORM OF BOTH SIDES AND USE THE IDENTITY:

$$\mathcal{L}(\dot{x}) = s\,\mathcal{L}(x) = s\,\overline{X}$$

$$\mathcal{L}(\ddot{x} + 2\omega_n \xi \dot{x} + \omega_n^2 x) = \mathcal{L}\left(\frac{1}{m} \cdot f(t)\right)$$

$$\left(s^2 + 2\omega_n \xi s + \omega_n^2\right) \overline{X}_{(s)} = \frac{1}{m} F_{(s)}$$

$$(s-a)(s-b)\,\overline{X}_{(s)} = \frac{1}{m} F_{(s)} = \frac{1}{m} \cdot \frac{1}{s}$$

where $a = -\omega_n \xi + \omega_d i$ and $b = -\omega_n \xi - \omega_d i$

$$\overline{X}(s) = \frac{1}{m} \cdot \frac{F_s}{(s-a)(s-b)} = \boxed{\frac{1}{ms(s-a)(s-b)} = \overline{X}(s)}$$

We FIND THE INVERSE LAPLACE OF $\overline{X}(s)$ (FROM CRC Sta Table of Math. 26th Ed.)

$$m\,x(t) = \frac{e^{at}}{a(a-b)} - \frac{e^{bt}}{b(a-b)} + \frac{1}{ab}$$

see last page

$$a - b = -\omega_n \xi + \omega_d i + \omega_n \xi + \omega_d i$$

$$= 2\omega_d i$$

$$a(a-b) = (-\omega_n \xi + \omega_d i)(2\omega_d i)$$

$$= -2\omega_n \omega_d \xi i - 2\omega_d^2$$

$$\frac{1}{a(a-b)} = \frac{1}{-2\omega_n \omega_d \xi i - 2\omega_d} \cdot \left(\frac{2\omega_n \omega_d \xi i - 2\omega_d}{2\omega_n \omega_d \xi i - 2\omega_d} \right)$$

multiply by complex conjugate to make denominator real

$$= \frac{2\omega_n \omega_d \xi i - 2\omega_d^2}{4\omega_n^2 \omega_d^2 \xi^2 + 4\omega_d^4} = Ai - B$$

similarly

$$\frac{1}{b(a-b)} = \frac{2\omega_n \omega_d \xi i + 2\omega_d^2}{4\omega_n^2 \omega_d^2 \xi^2 + 4\omega_d^4} = Ai + B$$

where the definition of A & B is apparent.

$$e^{at} = e^{(-\omega_n \xi + \omega_d i)t} = e^{-\omega_n \xi t}\left(\cos \omega_d t + i \sin \omega_d t\right)$$

$$= C + Di$$

$$e^{bt} = e^{(-\omega_n \xi - \omega_d i)t} = e^{-\omega_n \xi t}\left(\cos \omega_d t - i \sin \omega_d t\right)$$

$$= C - Di$$

where the definition of C & D is apparent.

$$\frac{e^{at}}{a(a-b)} = (C + Di)(Ai - B) = (CA - DB)i - (CB + DA)$$

$$\frac{e^{bt}}{b(a-b)} = (C - Di)(Ai + B) = (CA - DB)i + (CB + DA)$$

$$\frac{e^{at}}{a(a-b)} - \frac{e^{bt}}{b(a-b)} = -2(CB + DA)$$

$$C \cdot B = \left(e^{-\omega_n \xi t} \cos \omega_d t\right)\left(\frac{2\omega_d^2}{4\omega_n^2 \omega_d^2 \xi^2 + 4\omega_d^4}\right)$$

$$DA = \left(e^{-\omega_n \xi t} \sin \omega_d t\right)\left(\frac{2\omega_n \omega_d \xi}{4\omega_n^2 \omega_d^2 \xi^2 + 4\omega_d^4}\right)$$

$$-2(CB + DA) = -e^{-\omega_n \xi t}\left[\cos \omega_d t + \frac{\xi}{\sqrt{1-\xi^2}} \sin \omega_d t\right]\frac{1}{\omega_n^2}$$

Note that:

$$4\omega_n^2 \omega_d^2 \xi^2 + 4\omega_d^4 = 4(\underbrace{\omega_n^2 \omega_d^2}_{\omega_n^2(1-\xi^2)} \xi^2 + \underbrace{\omega_d^4}_{\omega_n^4(1-\xi^2)^2}) = 4\omega_n^4(1-\xi^2) = 4\omega_d^2 \omega_n^2$$

Finally

$$\frac{1}{ab} = \frac{1}{-\omega_n \xi + \omega_d i} \cdot \frac{1}{-\omega_n \xi - \omega_d i} = \frac{1}{\omega_n^2 \xi^2 + \omega_d^2}$$

$$= \frac{1}{\omega_n^2 \xi^2 + \omega_n^2 (1-\xi^2)} = \frac{1}{\omega_n^2}$$

Combining all terms:

$$x(t) = -\frac{1}{m\omega_n^2} e^{-\omega_n \xi t} \left[\cos \omega_d t + \frac{\xi}{\sqrt{1-\xi^2}} \sin \omega_d t \right]$$

$$+ \frac{1}{m\omega_n^2}$$

$$\boxed{x(t) = -\frac{1}{K} e^{-\omega_n \xi t} \left[\cos \omega_d t + \frac{\xi}{\sqrt{1-\xi^2}} \sin \omega_d t \right] + \frac{1}{K}}$$

recall $m\omega_n^2 = K$

Ex.

$m = 1$

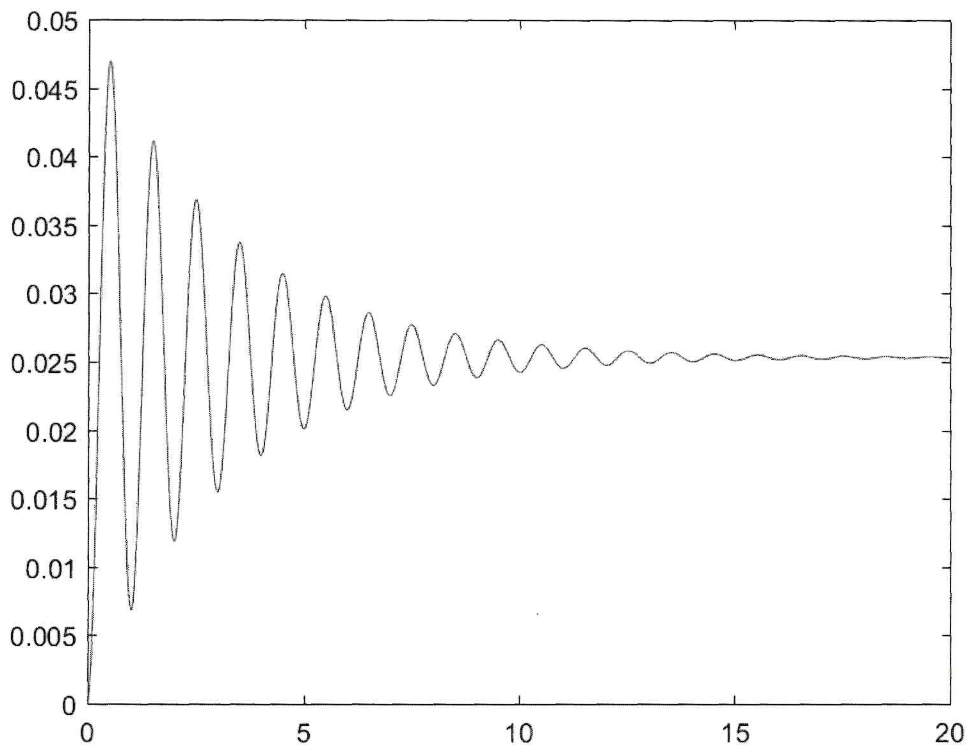
$\omega_n = 2\pi$

$\zeta = 0.05$

$k = 4\pi^2$

$\frac{1}{k} = 0.0253$

$x(t)$



time (s)

LAPLACE TRANSFORMS

	$f(s)$	$F(t)$
1	$\frac{1}{s}$	$\mu(t)$, unit step function
2	$\frac{1}{s^2}$	t
3	$\frac{1}{s^n}\ (n = 1, 2, \ldots)$	$\frac{t^{n-1}}{(n-1)!}$
4	$\frac{1}{\sqrt{s}}$	$\frac{1}{\sqrt{\pi t}}$
5	$s^{-3/2}$	$2\sqrt{\frac{t}{\pi}}$
6	$s^{-[n+(1/2)]}\ (n = 1, 2, \ldots)$	$\frac{2^n t^{n-(1/2)}}{1 \cdot 3 \cdot 5 \cdots (2n-1)\sqrt{\pi}}$
7	$\frac{\Gamma(k)}{s^k}\ (k > 0)$	t^{k-1}
8	$\frac{1}{s-a}$	e^{at}
9	$\frac{1}{(s-a)^2}$	te^{at}
10	$\frac{1}{(s-a)^n}\ (n = 1, 2, \ldots)$	$\frac{1}{(n-1)!} t^{n-1} e^{at}$
11	$\frac{\Gamma(k)}{(s-a)^k}\ (k > 0)$	$t^{k-1} e^{at}$
12*	$\frac{1}{(s-a)(s-b)}$	$\frac{1}{a-b}(e^{at} - e^{bt})$
13*	$\frac{s}{(s-a)(s-b)}$	$\frac{1}{a-b}(ae^{at} - be^{bt})$
14*	$\frac{1}{(s-a)(s-b)(s-c)}$	$-\frac{(b-c)e^{at} + (c-a)e^{bt} + (a-b)e^{ct}}{(a-b)(b-c)(c-a)}$
15	$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sin at$
16	$\frac{s}{s^2+a^2}$	$\cos at$
17	$\frac{1}{s^2-a^2}$	$\frac{1}{a}\sinh at$
18	$\frac{s}{s^2-a^2}$	$\cosh at$

*Here a, b, and (in 14) c represent distinct constants.