$$\int \frac{M\cdot \bar{m}\, dx}{EI} = \int \underbrace{\left(\frac{M}{EI}\right)}_{\text{any function}} \underbrace{\bar{m}}_{\text{linear (always)}}\, dx$$

$$\int \underbrace{f_{(x)}}_{\text{whatever}}(ax+b)\, dx =$$

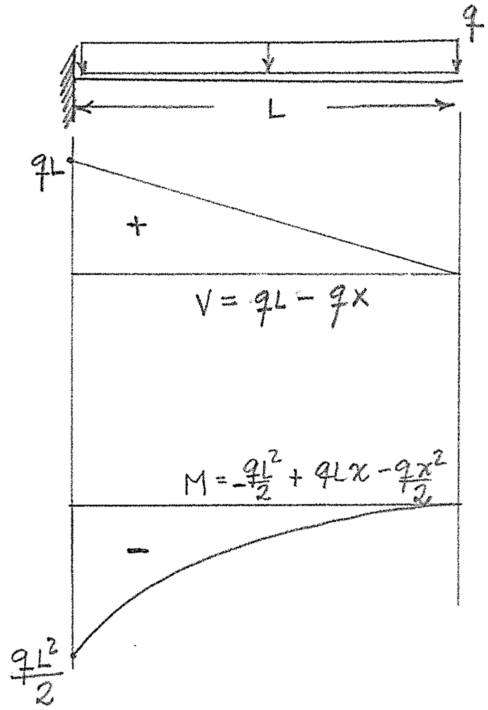
$$a\int f_{(x)}\cdot x\, dx + b\int f(x)\, dx =$$

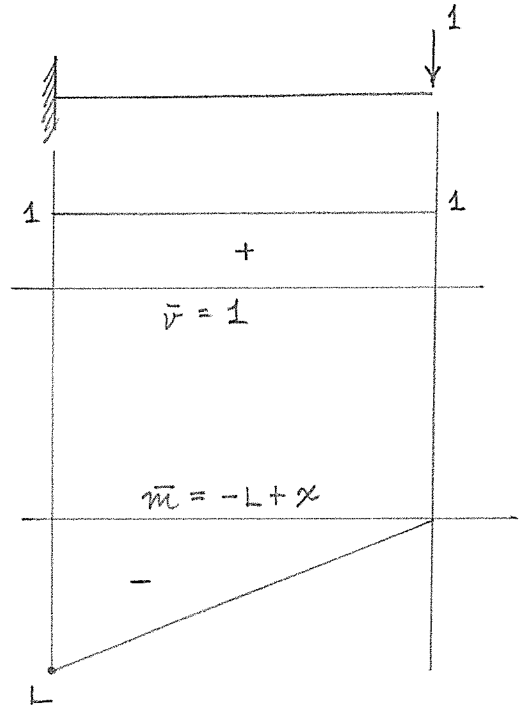
$$\left[a\underbrace{\frac{\int f(x)\cdot x\cdot dx}{\int f(x)\, dx}}_{\text{centroid of } f(x).} + b\right]\cdot \int f(x)\, dx$$

$$\underbrace{[a\cdot x_c + b]}_{\bar{m}\text{ @ centroid of }\frac{M}{EI}} \cdot A_f = \cancel{\int f(x)\cdot dx}$$

area under $\frac{M}{EI}$

$$\int \left(\frac{M}{EI}\right)\cdot \bar{m}\, dx = \left(\int \frac{M}{EI}\right)\cdot \bar{m}(x_c)$$





$$\int_0^L \frac{M\bar{m}}{EI}\,dx = \frac{1}{EI}\int_0^L \left(-\frac{qL^2}{2} + qLx - \frac{qx^2}{2}\right)(x-L)\,dx$$

$$= \frac{1}{EI}\int_0^L \left(-\frac{qL^2}{2}x + qLx^2 - \frac{qx^3}{2} + \frac{qL^3}{2} - qL^2x + \frac{qx^2L}{2}\right)dx$$

$$= \left[-\frac{qL^2x^2}{4} + \frac{qLx^3}{3} - \frac{qx^4}{8} + \frac{qL^3x}{2} - \frac{qL^2x^2}{2} + \frac{qLx^3}{6}\right]_0^L \frac{1}{EI}$$

$$\frac{1}{EI}\left[-\frac{qL^4}{4} + \frac{qL^4}{3} - \frac{qL^4}{8} + \cancel{\frac{qL^4}{2}} - \cancel{\frac{qL^4}{2}} + \frac{qL^4}{6}\right] = \boxed{\frac{qL^4}{8EI} = \Delta_M}$$

$$\int_0^L \frac{V\bar{v}\,dx}{KAG} = \int_0^L \frac{q(L-x)\cdot 1\cdot dx}{KAG} = \left[Lx - \frac{x^2}{2}\right]_0^L \cdot \frac{q}{KAG}$$

$$= \boxed{\frac{qL^2}{2KAG} = \Delta_V}$$

FOR RECTANGULAR CROSS SECTION

$$K = \frac{10(1+\nu)}{12+11\nu}$$

FOR CIRCULAR CROSS SECTION

$$K = \frac{6(1+\nu)}{7+6\nu}$$

ν = Poisson's ratio = 0.30 for steel

0.20 " Concrete

say we have a rectangular section

$A = b \cdot h$ $\quad G = \dfrac{E}{2(1+\nu)}$

$I = \dfrac{bh^3}{12}$

$$\Delta = \Delta_M + \Delta_V = \frac{qL^4}{8E \cdot \frac{1}{12} bh^3} + \frac{qL^2}{2K\,bh\, E/2(1+\nu)}$$

$$\Delta = \frac{3qL^4}{2bh^3E} + \frac{qL^2\, 2(1+\nu)}{2bhK\,E} = \frac{qL}{bE}\left[\frac{3}{2}\cdot\left(\frac{L}{h}\right)^3 + \left(\frac{1+\nu}{K}\right)\cdot\left(\frac{L}{h}\right)\right]$$

$$\Delta = \frac{qL}{bE}\left[\underbrace{\frac{3}{2}\left(\frac{L}{h}\right)^3}_{\text{BENDING}} + \overbrace{\frac{12+11\nu}{10}\cdot\left(\frac{L}{h}\right)}^{\text{SHEAR}}\right]$$

Using the fact that:

$$\int_0^L \frac{M\bar{m}}{EI}\,dx = A_{\frac{M}{EI}} \cdot \bar{m}(\bar{x}_{\frac{M}{EI}})$$

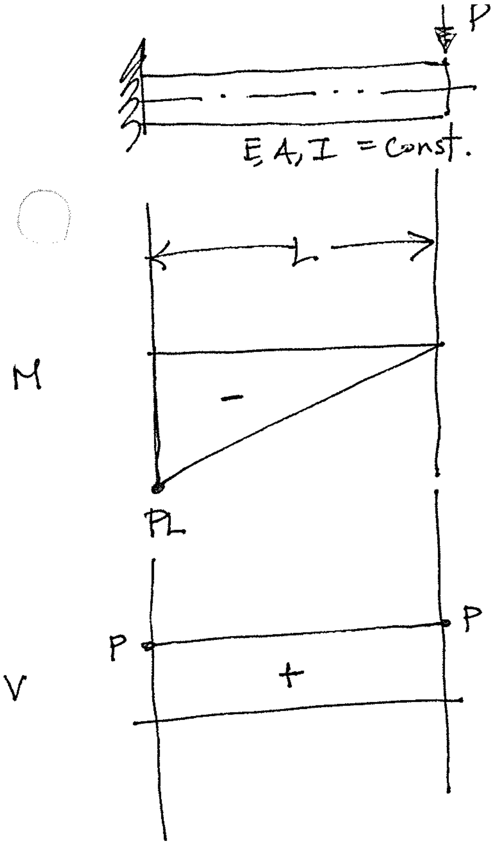
$$A_{\frac{M}{EI}} = \frac{1}{3} \cdot \frac{qL^2}{2} \cdot L = \frac{qL^3}{6EI}$$

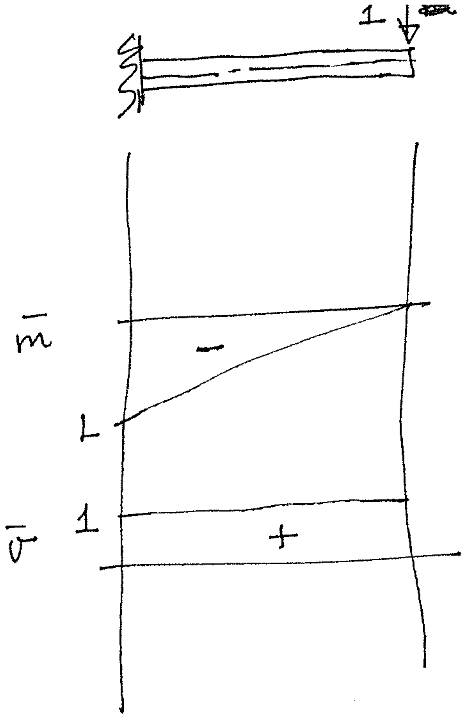
$$\bar{x}_{\frac{M}{EI}} = -L + \frac{L}{4} = \frac{3L}{4}$$

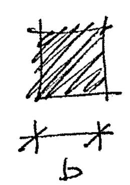
$$A_{\frac{M}{EI}} \cdot \bar{m}(\bar{x}_{\frac{M}{EI}}) = \frac{qL^3}{6EI} \cdot \frac{3L}{4} = \frac{qL^4}{8EI}$$

$$\boxed{\Delta_M = \frac{qL^4}{8EI}}$$

$$\int \frac{V\bar{v}\,dx}{KAG} = A_V \cdot \bar{v}(\bar{x}) = \frac{qL^2}{2} \cdot 1 \cdot \frac{1}{KAG} = \boxed{\frac{qL^2}{2KAG} = \Delta_V}$$







$K = 5/6$
for rectangular sections

$$G = \frac{E}{2(1+\nu)}$$

$$\Delta = \int_0^L \frac{M\bar{m}\,dx}{EI} + \int_0^L \frac{V\bar{v}\,dx}{KAG}$$

$$\Delta = \frac{1}{3}\cdot\frac{PL\cdot L\cdot L}{EI} + \frac{6P\cdot 1\cdot L}{5AG}$$

$$\Delta = \frac{PL^3}{3EI} + \frac{6PL}{5AG} = \frac{\overset{4}{\cancel{12}}PL^3}{\cancel{3}Eb^4} + \frac{6PL\,2(1+\nu)}{3b^2 E}$$

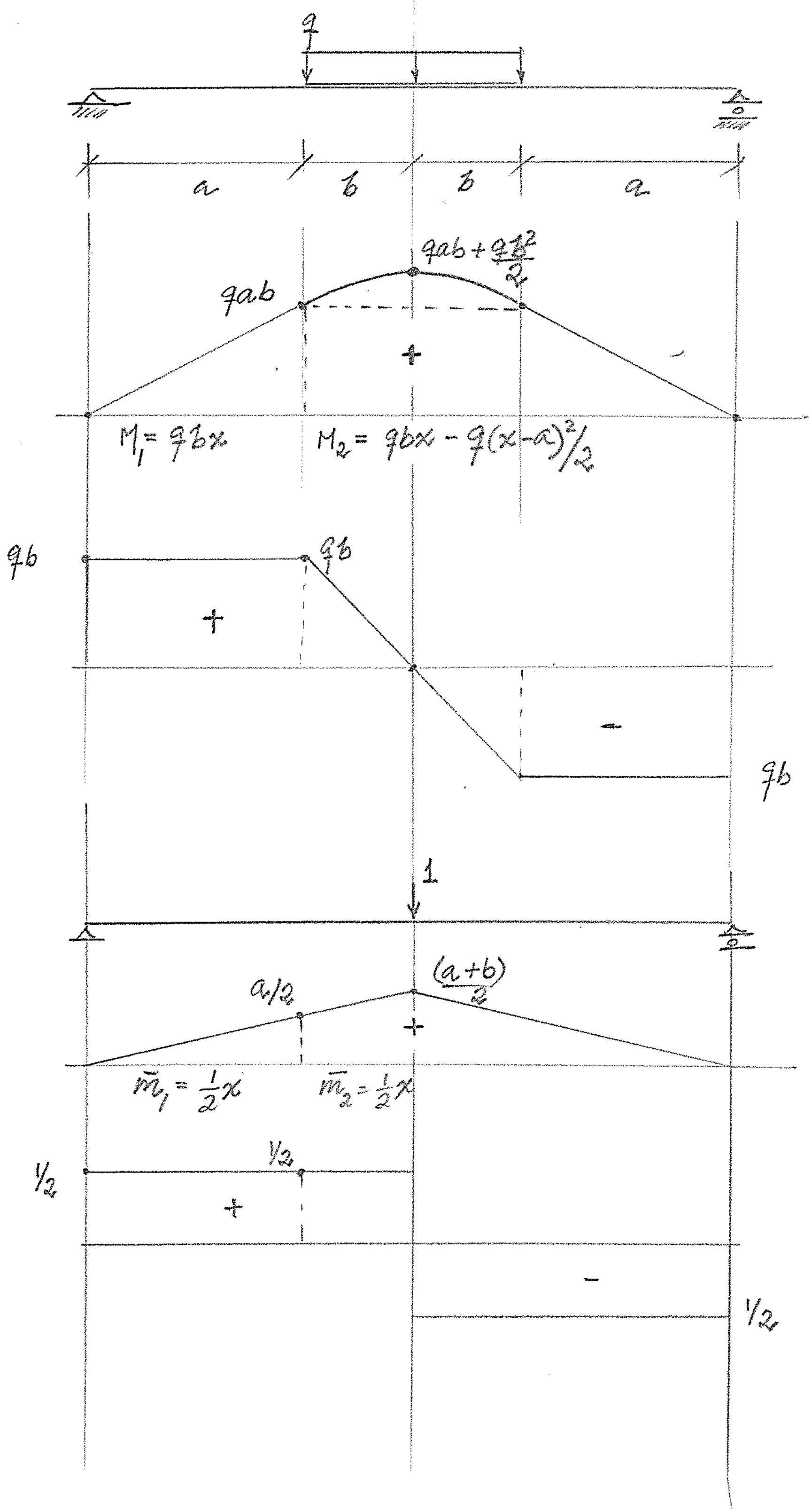
For Concrete $\nu \approx 0.17$	
$\frac{PL}{15Eb}\left[20\left(\frac{L}{b}\right)^3 + 42\left(\frac{L}{b}\right)\right]$	
for $\frac{L}{b} = 3$	for $\frac{L}{b} = 5$
$\frac{20\left(\frac{L}{b}\right)^3}{42\left(\frac{L}{b}\right)} \cong 4.29$	~ 12

$$\frac{4PL^3}{3Eb^4} + \frac{12\,PL(1+\nu)}{5Eb^2} = \left[20\left(\frac{L}{b}\right)^3 + 36\left(\frac{L(1+\nu)}{b}\right)\right]\frac{PL}{15Eb}$$

$$\left[20\left(\frac{L}{b}\right)^3 + 36(1+\nu)\left(\frac{L}{b}\right)\right]\frac{PL}{15Eb} = \Delta$$

Shear deformation can be neglected for $\frac{L}{b} > 5$ less than 10%

$EI = const.$


q
a
b
b
a
qab
$qab + \frac{qb^2}{2}$
+
$M_1 = qbx$
$M_2 = qbx - q(x-a)^2/2$
qb
qb
+
-
qb
1
$a/2$
$\frac{(a+b)}{2}$
+
$\bar{m}_1 = \frac{1}{2}x$
$\bar{m}_2 = \frac{1}{2}x$
$1/2$
$1/2$
+
-
$1/2$

$$\int \frac{M\bar{m}\,dx}{EI} = A_M\,\bar{m}(\bar{x})$$

for section 1.

$$A_{M_1} = \frac{1}{2}\,qab \cdot a = \frac{qa^2b}{2} \qquad \bar{x} = \frac{2a}{3}$$

$$\bar{m}(\bar{x}) = \frac{1}{2}\left(\frac{2a}{3}\right) = \frac{a}{3}$$

$$A_{M_1}\cdot \bar{m}(\bar{x}) = \frac{qa^3b}{6EI}$$

for section 2.

rect. parabola

$$A_{M_2} = qab \cdot b + \frac{b}{3}\cdot\left(\cancel{qab} + \frac{qb^2}{2} - \cancel{qab}\right)\cdot b$$

$$\Sigma A_{M_{2i}}\cdot m(\bar{x}_i) = qab^2\cdot \frac{1}{2}\left(a + \frac{b}{2}\right) + \frac{qb^3}{3}\cdot\frac{1}{2}\left(a + \frac{5}{8}b\right)$$

$$= \frac{qa^2b^2}{2} + \frac{qab^3}{4} + \frac{qab^3}{6} + \frac{5qb^4}{48}$$

$$= \frac{qa^2b^2}{2EI} + \frac{5qab^3}{12EI} + \frac{5qb^4}{48EI}$$

$$\Delta = 2\left[\frac{qa^3b}{6EI} + \frac{qa^2b^2}{2EI} + \frac{5qab^3}{12EI} + \frac{5qb^4}{48EI}\right]$$

$$\Delta = \int_0^L \frac{M\bar{m}}{EI}dx + \cancel{\int_0^L \frac{V\bar{v}\,dx}{KAG}}$$

$$\Delta = \int_0^a \frac{(qbx)(\frac{1}{2}x)}{EI}dx + \int_a^{b+a}\left(qbx - \frac{q(x-a)^2}{2}\right)\left(\frac{1}{2}x\right)\frac{dx}{EI}$$

$$\Delta = \frac{qbx^3}{6EI}\Big|_0^a + \frac{qbx^3}{6EI}\Big|_a^{b+a} - \left(\frac{qx^4}{16EI} - \frac{qax^3}{6} + \frac{qa^2x^2}{8}\right)\Big|_a^{b+a}$$

$$\Delta = \cancel{\frac{qa^3b}{6EI}} + \frac{qb(a+b)^3}{6EI} - \cancel{\frac{qa^3b}{6EI}}$$

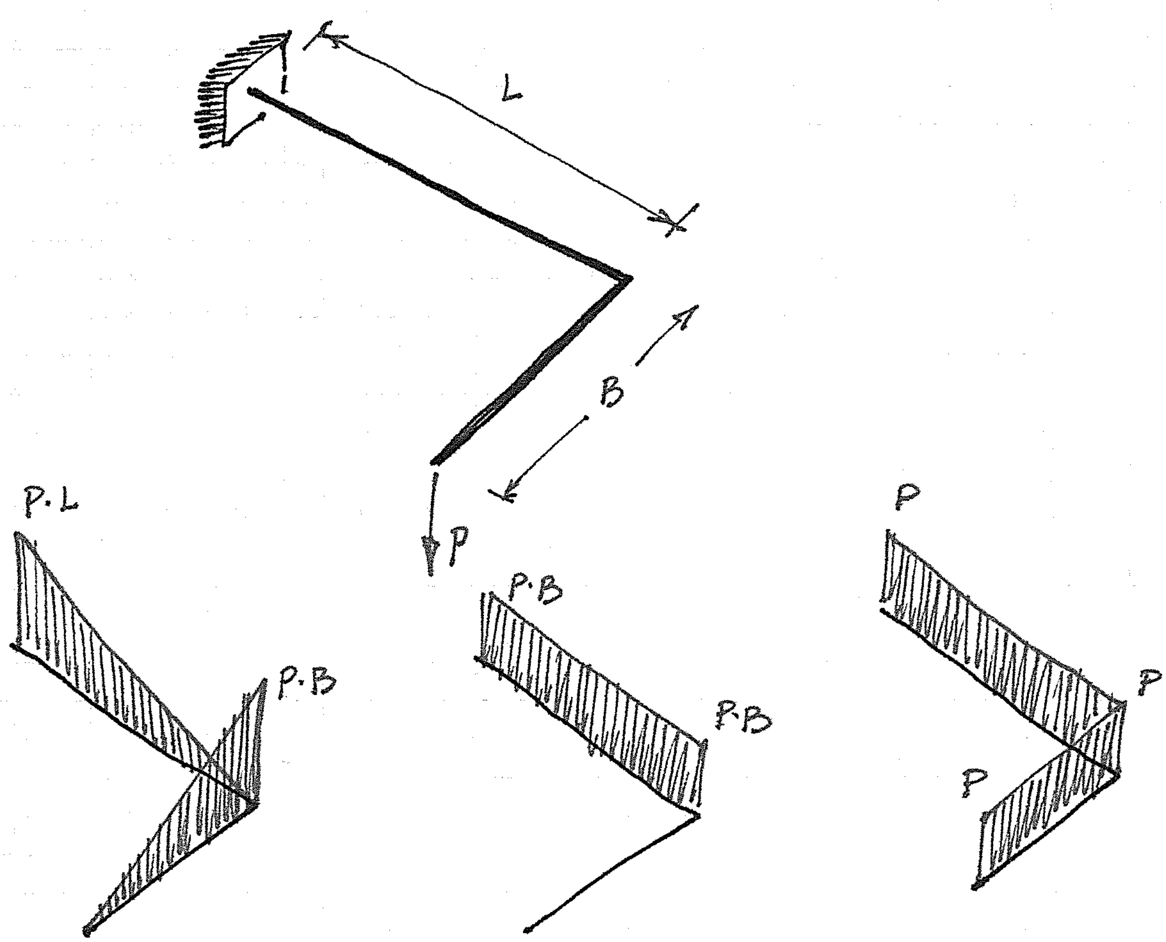
$$- \frac{q(a+b)^4}{16EI} + \frac{qa(a+b)^3}{6EI} - \frac{qa^2(a+b)^2}{8EI}$$

$$\underbrace{\boxed{+\frac{qa^4}{16EI} - \frac{qa^4}{6EI} + \frac{qa^4}{8EI}}}_{\frac{qa^4}{48}}$$

$$\Delta = 2\left[\frac{qb(a+b)^3}{6EI} - \frac{q(a+b)^4}{16EI} + \frac{qa(a+b)^3}{6EI} - \frac{qa^2(a+b)^2}{8EI} + \frac{qa^4}{48EI}\right]$$

Virtual Work to Compute deformations in 3-D frame structures.

$$\delta = \underbrace{\int \frac{N\bar{n}\,dx}{AE}}_{\text{Axial}} + \underbrace{\int \frac{V\bar{v}\,dx}{a_r G}}_{\text{Shear}} + \underbrace{\int \frac{M\bar{m}\,dx}{EI}}_{\text{Bending Moment}} + \underbrace{\int \frac{T\bar{t}\,dx}{GJ}}_{\text{Torsion}}$$



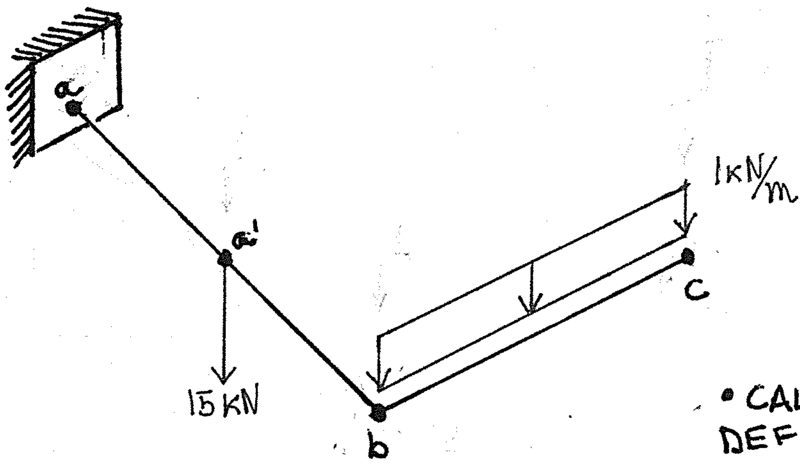
Bending Moment　　Torsion　　Shear

$$\int \frac{N\bar{n}\,dx}{AE} = 0$$

$$\int \frac{V\bar{v}\,dx}{a_r G} = \frac{(P\cdot L)(1)}{a_r G} + \frac{(P\cdot B)(1)}{a_r G}$$

$$\int \frac{M\bar{m}\,dx}{EI} = \left(\frac{1}{2}PL^2\right)\cdot\left(\frac{2}{3}L\right)\frac{1}{EI} + \left(\frac{1}{2}PB^2\right)\left(\frac{2}{3}B\right)\frac{1}{EI}$$

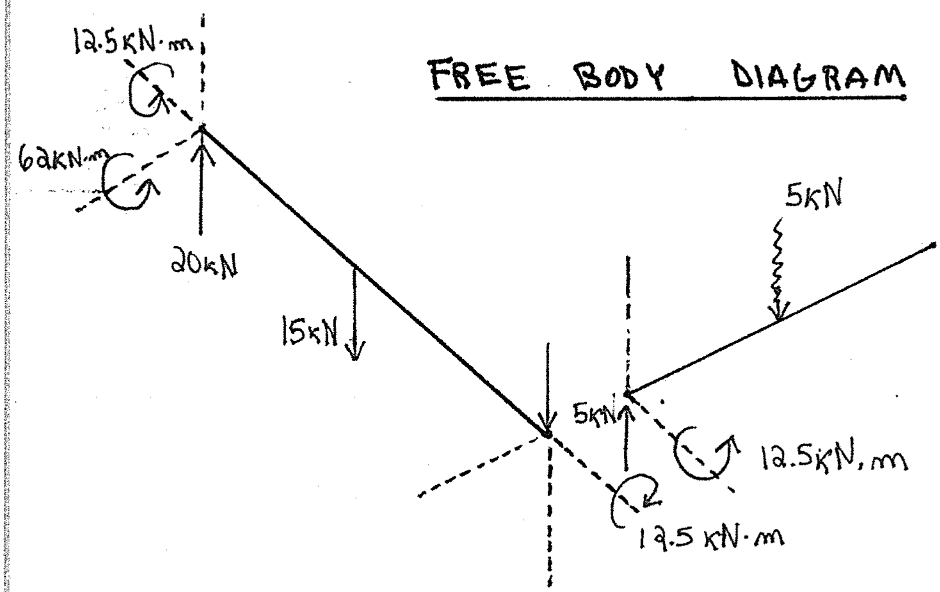
$$\int \frac{T\bar{t}}{GJ}\,dx = (P\cdot B\cdot L)(B)\frac{1}{GJ}$$



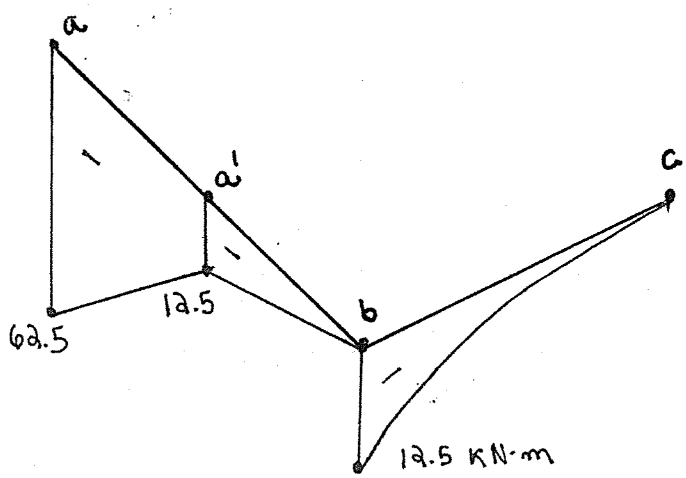
a - a' = 2.5 m
a - b = 5.0 m
b - c = 5.0 m

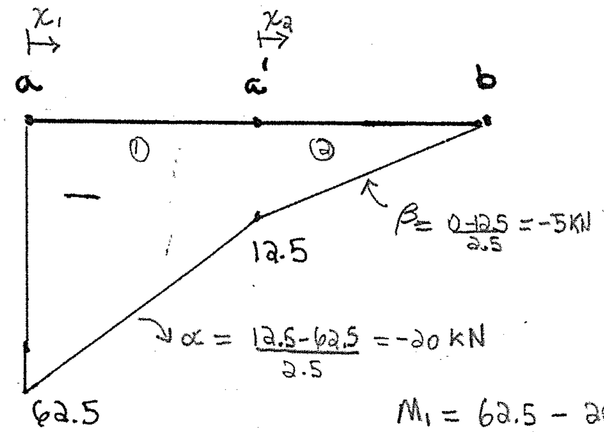
- CALCULATE VERTICAL DEFORMATION @ a', b, c

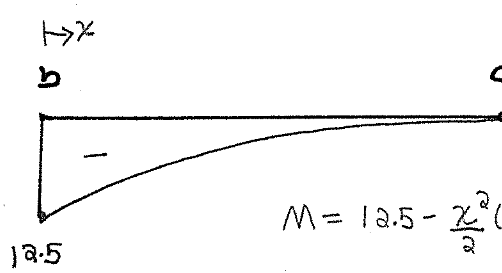
FREE BODY DIAGRAM



BENDING MOMENT (KN·m)





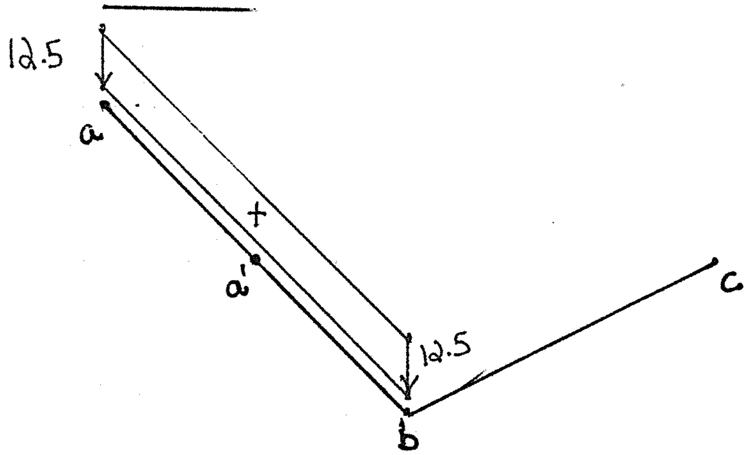


$$M = 12.5 - \frac{x^2}{2}(1)$$

$$M_1 = 62.5 - 20x_1 \qquad 0 \le x_1 \le 2.5$$

$$M_2 = 12.5 - 5x_2 \qquad 0 \le x_2 \le 2.5$$

TORSION (KN·m)

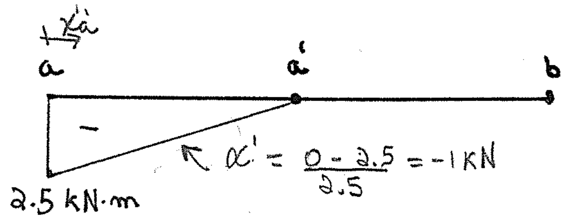


$$T_1 = 12.5 \qquad \forall x \quad a-b$$

$$T_2 = 0 \qquad \forall x \quad b-c$$

$$\delta_{vi} = \int_a^c \frac{M\bar{m}}{EI}\,dx + \int_a^c \frac{T\bar{t}}{GJ}\,dx$$

$\delta_{va'} \rightarrow$ UNIT LOAD @ a'



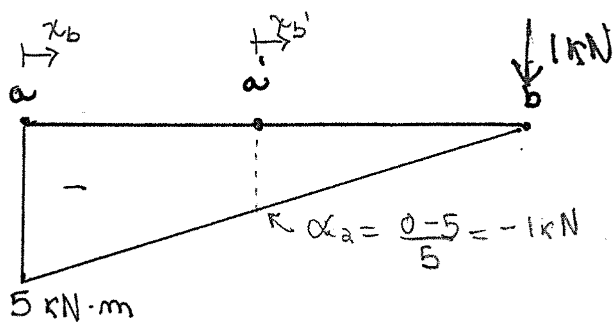
$\bar{m}_{va'} = 2.5 - x_{a'}$ $\qquad 0 \le x_{a'} \le 2.5$

$$\delta_{va'} = \frac{1}{EI}\int_0^{2.5}(62.5-20x)(2.5-x)\,dx = \frac{1}{EI}\int_0^{2.5}(156.25-62.5x-50x+20x^2)\,dx$$

$$= \frac{1}{EI}\int_0^{2.5}(156.25-112.5x+20x^2)\,dx = \frac{1}{EI}\left[156.25x - 112.5\frac{x^2}{2} + \frac{20x^3}{3}\right]\Bigg|_0^{5}$$

$$= \frac{1}{EI}\left[143.2292\right]$$

$\delta_{vb} \Rightarrow$ UNIT LOAD @ b



$\bar{m}_{vb} = 5 - x_b$ $\qquad 0 \le x_b \le 2.5$

$\bar{m}_{vb'} = 2.5 - x_{b'}$ $\qquad 0 \le x_{b'} \le 2.5$

$$\delta_b = \frac{1}{EI}\int_0^{5} M\bar{m}_b\,dx = \frac{1}{EI}\int_0^{2.5}(62.5-20x)(5-x)\,dx + \frac{1}{EI}\int_0^{2.5}(12.5-5x)(2.5-x)\,dx$$

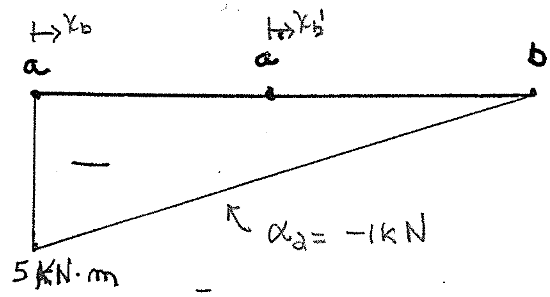
$$\delta_{vb} = \frac{1}{EI}\int_0^{2.5}(312.5 - 162.5x + 20x^2)\,dx + \frac{1}{EI}\int_0^{2.5}(31.25 - 25x + 5x^2)\,dx$$

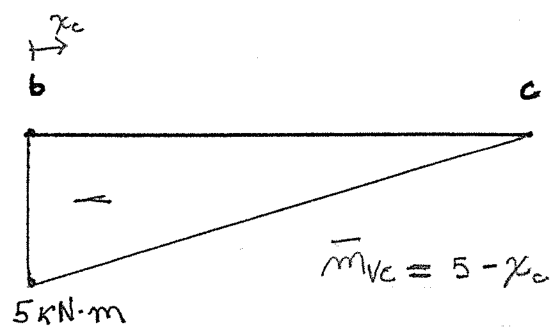
$$= \frac{1}{EI}\left[312.5x - 162.5\frac{x^2}{2} + 20\frac{x^3}{3}\right]\Bigg|_0^{2.5} + \frac{1}{EI}\left[31.25x - 25\frac{x^2}{2} + 5\frac{x^3}{3}\right]\Bigg|_0^{2.5}$$

$$= \frac{1}{EI}\left[377.6042\right] + \frac{1}{EI}\left[26.0417\right]$$

$$\delta_{vb} = \frac{1}{EI}\left[403.6459\right]$$

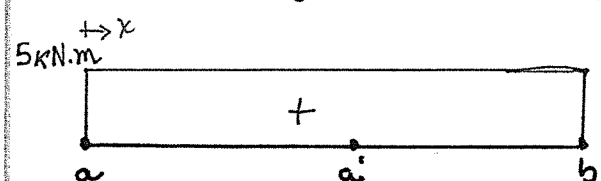
$\delta_{vc} \rightarrow$ UNIT LOAD @ C





$\bar{m}_{vb} = 5 - x_b \qquad 0 \le x_b \le 2.5 \qquad\qquad 0 \le x_c \le 5$

$\bar{m}_{vb'} = 2.5 - x_b' \qquad 0 \le x_b' \le 2.5$



$\bar{t}_{ab} = 5 \qquad \forall\, x\ [a\ b]$

$$\delta_{cv} = \frac{1}{EI}\int_a^b M\bar{m}\,dx + \frac{1}{EI}\int_b^c M\bar{m}\,dx + \frac{1}{GJ}\int_a^b T\bar{t}\,dx$$

$$= \delta_{vb} + \frac{1}{EI}\int_0^5\left(12.5 - \frac{x^2}{2}\right)(5 - x)\,dx + \frac{1}{GJ}\int_0^5 (12.5)(5)\,dx$$

$$= \delta_{vb} + \frac{1}{EI}\int_0^5\left(62.5x - 12.5x - 5\frac{x^2}{2} + \frac{x^3}{2}\right)dx + \frac{1}{GJ}\int_0^5 62.5\,dx$$

$$\delta v_c = \delta v_b + \frac{1}{EI}\left(62.5x - 12.5\frac{x^2}{2} - 5\frac{x^3}{6} + \frac{x^4}{8}\right)\Big|_0^5 + \frac{1}{GJ}\left(62.5x\right)\Big|_0^5$$

$$= \frac{1}{EI}[403.6459] + \frac{1}{EI}[206.6458] + \frac{1}{GJ}[312.5]$$

$$\delta v_c = \frac{1}{EI}[610.2917] + \frac{1}{GJ}[312.5]$$

SUMMARY

$$\delta v_a' = \frac{1}{EI}(143.2292)$$

$$\delta v_b = \frac{1}{EI}(403.6459)$$

$$\delta v_c = \frac{1}{EI}(610.2917) + \frac{1}{GJ}[312.5]$$

EVALUATION OF E, I, G, J

$E = 200\ GPa = 2 \times 10^{11}\ Pa = 2 \times 10^8\ KN/m^2$

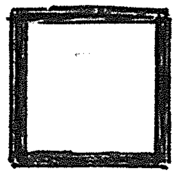
$G = \frac{E}{2(1+\nu)} = \frac{2 \times 10^8}{2(1+0.3)} = 7.69230769 \times 10^7\ KN/m^2$

W18X60 $\quad I = 984\ IN^4 = 4.0957172 \times 10^{-4}\ m^4 \quad A = 17.6\ IN^2$

$J = 2.17\ IN^4 = 9.0322219 \times 10^{-7}\ m^4$

$\delta v_a' = 0.001749\ m = 0.1749\ cm \qquad \delta v_c = 4.5052\ m$

$\delta v_b = 0.004927\ m = 0.4927\ cm$



HSS 10X10X1/2

$I = 256\ IN^4 = 1.065555 \times 10^{-4}\ m^4$

$J = 412\ IN^4 = 1.714873 \times 10^{-4}\ m^4$

$A = 17.2 IN^2$

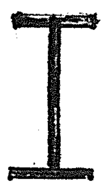
$W = 62.46\ lb/ft$

$\delta v a' = 0.006721\ m = 0.6721\ cm$

$\delta v b = 0.018941\ m = 1.8941\ cm$

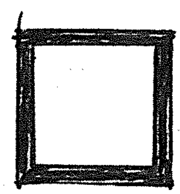
$\delta v c = 0.052327\ m = 5.2327\ cm$

HYBRID



b-c

W18x60



a-b

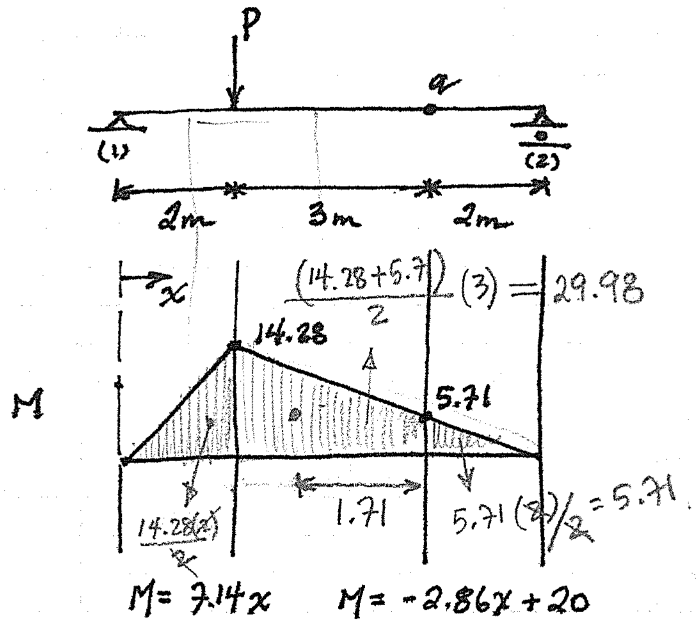
HSS 10X10X1/2

$\delta v a' = 0.006721\ m = 0.6721\ cm$

$\delta v b = 0.01841\ m = 1.8941\ cm$

$\delta v c = 0.045153\ m = 4.5153\ cm$

VIRTUAL WORK - COMPUTE DEFORMATIONS.



$\delta_a =$

$EI = \text{const.}$

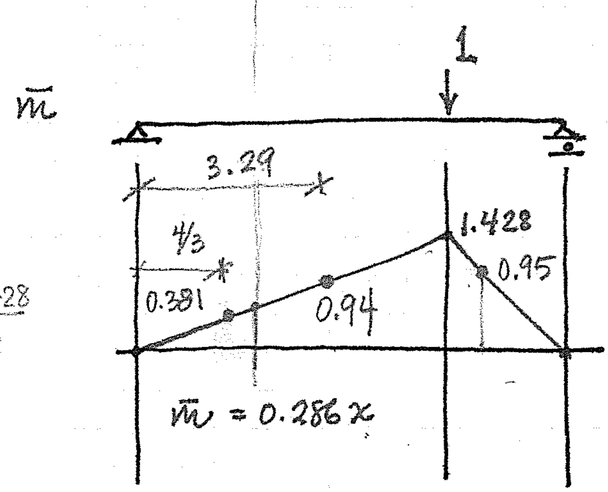
$P = 10\text{ kN.}$

$R_1 = \frac{5P}{7} = 0.71(10) = 7.14\text{ kN}$

$R_2 = \frac{2P}{7} = 2.86\text{ kN}$

$ax+b$

$\int \frac{M\bar{m}\,dx}{EI} = A_M \cdot \bar{m}(\bar{x}_M)$



$\frac{\bar{m}}{(4/3)} = \frac{1.428}{5}$

$R_1 = 0.286$

$R_2 = 0.714$

$\frac{1.428}{2} \cdot \left(\frac{4}{3}\right) =$

$\bar{m} = -0.714x + 5$

$$EI\delta = \int_0^2 (7.14x)(0.286x)\,dx + \int_2^5 (-2.86x+20)(0.286x)\,dx$$

$(14.28)(0.381)$ 5.44 ✓ $(29.98)(0.94)$ 28.12 ✓

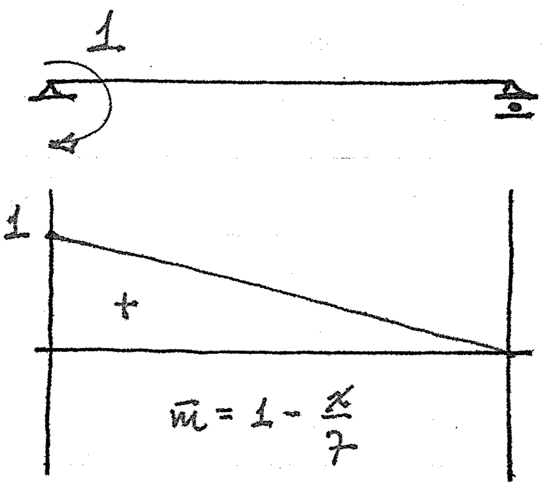
$$+\int_5^7 (-2.86x+20)(-0.714x+5)\,dx$$

$(5.71)\,(0.95) =$

5.42 ✓

5.445 ✓ $+ 28.16$ ✓ $+ 5.419$ ✓ $= 39.02$

Find rotation at support:



$R_1 = -0.143$

$R_2 = 0.143$

$$EI\,\theta = \int_0^2 (7.14x)\left(1 - \frac{x}{7}\right)dx + \int_2^7 (-2.86x + 20)\left(1 - \frac{x}{7}\right)dx$$

$$= A_M \cdot \bar{m}(\bar{x}_M) + A_M\, \bar{m}(\bar{x}_M)$$

$$= 14.28(0.81) + 35.7(0.476)$$

$$= 11.57 + 17.00$$

$$11.56 + 17.00$$