CE-370 Uncertainty and Risk in Engineering Systems

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SUMMARY OF LECTURE 6 - FUNCTIONS OF A RANDOM VARIABLE

In many situations one is interested in determining the cumulative distribution function (CDF) and probability density function (PDF) of a certain function of a random variable with known CDF and PDF. That is, given $F_X(x), f_X(x)$ and the function

$$Y = \phi(X) \tag{1}$$

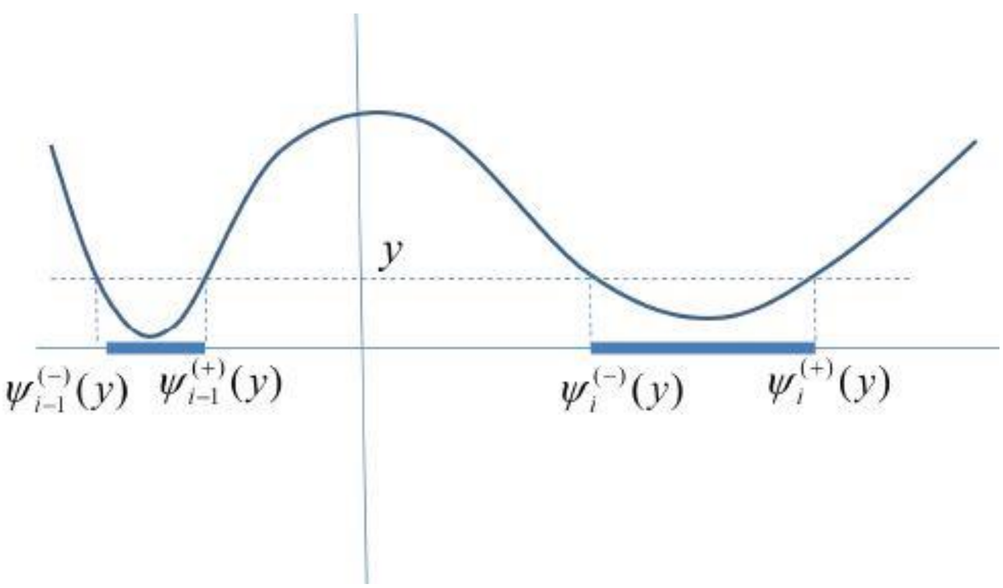
we are interested in

$$F_Y(y) = P[Y \leq y] \tag{2}$$

and its derrivative

$$f_Y(y) = \frac{d}{dy}F_Y(y) \tag{3}$$

To begin, consider the figure below depicting an arbitrary function $y = \phi(x)$



As can be seen, the event $Y \leq y$ is equivalent to the event $\bigcup_i \psi_i^{(-)}(y) \leq x \leq \psi_i^{(+)}(y)$ where $\psi(\cdot) = \phi^{-1}(\cdot)$. Therefore

$$F_Y(y) = P[Y \leq y] \Leftrightarrow \sum_i P[\psi_i^{(-)} \leq x \leq \psi_i^{(+)}(y)] = \sum_i \big(F_X(\psi_i^{(+)}(y)) - F_X(\psi_i^{(-)}(y))\big) \tag{4}$$

Consequently, the probability density function (PDF) of Y is given by

$$f_Y(y) = \frac{df_y(y)}{dy} = \sum_i \frac{d\big(F_X(\psi_i^{(+)}(y)) - F_X(\psi_i^{(-)}(y))\big)}{dy} \tag{5}$$

and using the chain rule of differentiation we obtain a general expression for $f_Y(y)$

$$f_Y(y) = \sum_i \bigg(\frac{d\psi_i^{(+)}(y)}{dy} f_X(\psi_i^{(+)}(y)) - \frac{d\psi_i^{(-)}(y)}{dy} f_X(\psi_i^{(-)}(y))\bigg) \tag{6}$$

SPECIAL CASE 1: MONOTONICALLY INCREASING FUNCTION If $\phi(x)$ increases monotonically, the results in eq. 6 and 4 simplify to

$$F_Y(y) = F_X(\psi(y)) - F_X(-\infty) = F_X(\psi(y)) \tag{7}$$

$$f_Y(y) = \frac{d\psi(y)}{dy} f_X(\psi(y)) \tag{8}$$

SPECIAL CASE 2: MONOTONICALLY DECREASING FUNCTION If $\phi(x)$ decreases monotonically, the results in eq. 6 and 4 simplify to

$$F_Y(y) = F_X(+\infty) - F_X(\psi(y)) = -F_X(\psi(y)) \tag{9}$$

$$f_Y(y) = -\frac{d\psi(y)}{dy} f_X(\psi(y)) \tag{10}$$

Note that since the function is monotonically decreasing, the derivative is negative and thus

$$f_Y(y) = |\frac{d\psi(y)}{dy}| f_X(\psi(y)) \tag{11}$$

which is actually an expression for any monotonic function.

EXPECTED VALUE OF FUNCTIONS OF RANDOM VARIABLES

The expected value of a function $y = \phi(x)$ of a random variable X is given by

$$E[Y] = \int_{-\infty}^{+\infty} \phi(x) f_X(x) dx \neq \phi(E[X]) \tag{12}$$

This expression is valid for any arbitrary function $\phi(x)$ monotonic or not.

VARIANCE OF FUNCTIONS OF RANDOM VARIABLES

By definition

$$VAR[Y] = E[Y^2] - E[Y]^2 \tag{13}$$

where

$$E[Y^2] = \int_{-\infty}^{+\infty} \phi^2(x) f_X(x) dx \tag{14}$$