CE-370 Uncertainty and Risk in Engineering Systemss

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SUMMARY OF LECTURE 4 - RANDOM VARIABLES

Random Variable

Let $(\Omega, \mathcal{F}, P)$ be a probability space, then a random variable is a invertible function X that maps every element of the sample space Ω (every possible event of the random experiment) to subsets of a metric space S, typically $\mathbb{R}$. Consider as examples:

- Tossing a coin. The elements of the sample space Ω are Heads(H) or Tails(T), so $\Omega = \{H, T\}$. The σ-field $\mathcal{F} = \{\emptyset, H, T, HT\}$. One possible random variable for this case could be a function X s.t. $\{(H, 0), (T, 2)\}$.
- Playing the wheel of fortune. In this case the elements of the sample Ω are the various prizes that appear in various sectors of the wheel of fortune. The corresponding σ-field $\mathcal{F}$ can be constructed by sequentially constructing unions and complements of the elements of Ω and the probability measure can be constructed assuming all outcomes are equiprobable. One possible random variable X for this case would be a function mapping every element of Ω to subsets of the interval $[0 \;\; 2\pi)$.

For most practical engineering applications, there are essentially two types of random variables: continuous and discrete, with the possibility of having mixed (continuous-discrete) random variables.

Continuous Random Variable: The range is defined as any subset of $\mathbb{R}$.

Discrete Random Variable: The range is defined by a finite number of values or at most, infinitely countable values from a set, typically, but not necessarely, the integers $\mathbb{Z}$.

Mixed Random Variable: The range is defined as a combination of discrete and continuous random variables. Consider tossing a coin, if the outcome is heads, a number between [0 0.5) is selected, otherwise a number between [0.5 1] is selected.

It is important to realize that some random variables do not fit into the above categories. Consider a random variable X which is the result of taking the ratio of two discrete random variables defined over the natural numbers $\mathbb{N}$. Clearly, X is defined only over the rational numbers $\mathbb{Q}$, which are a dense set, not continuous, but countable.

Cumulative Distribution Function

Let $(\Omega, \mathcal{F}, P)$ be a probability space and a random variable X, the the cumulative distribution function (CDF) $F_X(x)$ of a random variable X defines a probability measure on subsets of $\mathbb{R}$ induced by P. Some important properties of CDF derived from the basic axioms of probability (see Lecture Notes 2) are:

- The probability that a random variable lies in the closed interval $[a \;\; b]$ is given by: $P(a \leq X \leq b) = F_X^-(b) - F_X^+(a)$
- The probability that a random variable lies in the half-open interval $[a \;\; b)$ is given by: $P(a \leq X < b) = F_X^-(b) - F_X^-(a)$
- The probability that a random variable lies in the open interval $(a \;\; b)$ is given by: $P(a < X < b) = F_X^-(b) - F_X^+(a)$
- $P(X \leq x) = F_X(x)$. This results from the fact that $P(X \leq -\infty) = 0$.
- The CDF of a random variable is a non-decreasing function. In other words, $F_X(b) \geq F_X(a)$ if $b \geq a$. This is because the set represented by $-\infty \leq x \leq a$ is a subset of the set represented by $-\infty \leq x \leq b$.
- The CDF of a random variable must always lie between zero and one (inclusive), $0 \leq F_X(x) \leq 1$. This a clear consequence of the definition of a probability measure (see Lecture Notes 2).

Note that probability is only defined over sets (intervals of $\mathbb{R}$) and thus by consequence, $P(X = x) = 0$.

Probability Density Function

In many contexts it is more mathematically convenient to deal with the probability density function of a random variable than with its CDF. The probability density function (PDF) is defined as the derivative of the Cumulative Distribution Function (CDF) $F_X(x)$, that is:

$$f_X(x) = \frac{dF_X(x)}{dx}$$

Therefore, by definition

$$p(X \leq x) = F_X(x) = \int_{-\infty}^{x} f_X(x)dx$$

Some important properties of PDF derived from the properties of the CDF are:

- The probability that a random variable (X) lies in the interval defined by x=b and x=a is given by
$$p(b \leq X \leq a) = F_X(a) - F_X(b) = \int_{b}^{a} f_X(x)dx$$
- The integral of the PDF is unity (normalization condition), $\int_{-\infty}^{+\infty} f_X(x)dx = 1$

Extension to Discrete Random Variables

The concepts of CDF and PDF can be easily extended to discrete random variables by means of two complementary functions or more properly, distributions. A discussion regarding the mathematical subtleties involved in distinguishing between a function and distribution are beyond the scope of this course. The reader is encouraged to read Distribution Theory and Transform Analysis by Zemanian. These functions (or distributions) are the Heaviside function and the Dirac delta function.

Heaviside function: The Heaviside[1] function is defined as

$$H(t, t_0) = 1 \quad t \geq t_0$$

and

$$H(t, t_0) = 0 \quad t < t_0$$

Dirac-δ function: The Dirac[2]-δ function is defined as

$$\int_{-\infty}^{t} \delta(t, t_0) dt = 1 \quad t \geq t_0$$

and

$$\int_{-\infty}^{t} \delta(t, t_0) dt = 0 \quad t < t_0$$

As can be seen, the following relationship holds:

$$H(t, t_0) = \int_{-\infty}^{t} \delta(t, t_0) dt$$

Therefore for a discrete random variable with a sample space of n, the CDF can be expressed as

$$F_X(x) = \sum_{k=1}^{n} H(x, x_k) P(X = x_k)$$

It is evident that since $\sum P(x_k) = 1$, then $\int F_X(x) dx = 1$. Consequently, the PDF of a discrete random variable X can be expressed as

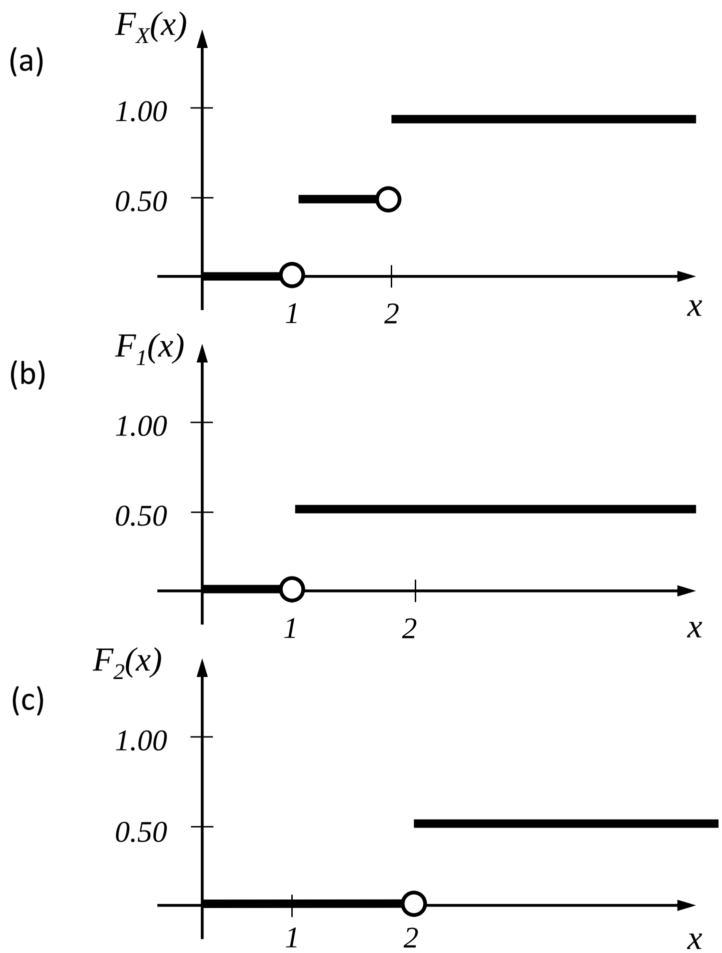
$$f_X(x) = \sum_{k=1}^{n} \delta(x, x_k) P(X = x_k)$$

EXAMPLE: Consider a random variable with sample space 1,2 and each value is equiprobable ($p(x_i) = 0.5$). The CDF of such a variable is shown in the figures below. Figure (a) shows the resulting CDF. Figures (b) and (c) depict the elementary functions (not CDF themselves) corresponding to each element in the sample space. Thus

$$F_X(x) = F_1(x) + F_2(x) = 0.50H(x, 1) + 0.50H(x, 2)$$

In this case the PDF is simply the superposition of two delta functions (not shown), one centered at 1 and another at 2.

[1] named after the mathematician Oliver Heaviside, 1850-1925
[2] named after the mathematician and physicist Paul Dirac, 1902-1984


(a)
$F_X(x)$
1.00
0.50
1
2
x
(b)
$F_1(x)$
1.00
0.50
1
2
x
(c)
$F_2(x)$
1.00
0.50
1
2
x