CE-370 Uncertainty and Risk in Engineering Systems

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SUMMARY OF LECTURE 3 - FOUNDATIONS OF PROBABILITY THEORY

Definitions

- *Random Experiment*: A random experiment is an experiment in which the outcome varies in an unpredictable fashion when the experiment is repeated under the "same conditions".
- *Probability Space*: A probability space $(\Omega, \mathcal{F}, P)$ consists of a sample space Ω, a σ-field $\mathcal{F}$ and a probability measure P.
- *Sample Space*: The sample space Ω of a random experiment is defined as the set of all possible outcomes. A sample space can be finite or infinite, countable or uncountable.
- *Event*: A subset of Ω
- *σ-field*: A collection of sets $\mathcal{F} \subset \Omega$ is a σ-field if: *(i)* $\emptyset \in \mathcal{F}$, *(ii)* if $A \in \mathcal{F}$ implies $A^c \in \mathcal{F}$ and *(iii)* if $A_i \in \mathcal{F}$, then $\cup_{i \in I} A_i \in \mathcal{F}$ where I is a countable indexing set. The pair $(\Omega, \mathcal{F})$ is a measurable space.
- *Basic Set Operations*:
  - Union of sets: Defined as the set of all of the elements that are members of at least one of the sets considered. For two sets A and B, the union of the two sets is represented as $A \cup B$.
  - Intersection of sets: Defined as the set of all elements that are members of all the sets considered. For two sets A and B, the intersection of the two sets is represented as $A \cap B$. If two sets do not have any elements in common, then we write $A \cap B = \emptyset$ and we say the sets are disjoint.
  - Complementary set: For a given set A, its complement is represented as A^c and it consists of all the elements of Ω not members of A.
  - Subset: A subset is a set of elements which are all contained within a greater (or equal) set. If A is a subset of B, then $A \subset B$, i.e. every member of A is also a member of B. The converse is true only if $A = B$.
- *Measure Space*: Let $(\Omega, \mathcal{F})$ be a measurable space. A set function $\mu : \mathcal{F} \rightarrow [0, \infty]$ such that: *(i)* $\mu(\emptyset) = 0$, *(ii)* if the sets A_i are mutually disjoint, then $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$
- *Probability Measure*: If $\mu(\Omega) < \infty$, then the measure space can be normalized such that $\mu : \mathcal{F} \rightarrow [0, 1]$. Any such measure is a probability measure P on $(\Omega, \mathcal{F})$.

Important corollaries

Below are some important corollaries that can be derived from the previous definitions:

- $P(A^c) = 1 - P(A)$
- $P(A) \leq 1$
- If $A \subseteq B$ $P(A) \leq P(B)$
- $P(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

In all above cases $A, B \in \mathcal{F}$. Also, from De Morgan's Laws we have that:

- $\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} A_i^c$
- $\left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} A_i^c$

where I is some indexing set.

Total Probability Theorem

If Ω is partitioned into n disjoint events $(B_1, B_2, ..., B_n)$, then

$$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$$

Conditional Probability Theorem

The probability of the simultaneous occurrence of two events A and B can be expressed as a function of their conditional probability as

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

where $P(A|B)$ is the probability of A *if* B were to occur. Note that this is a hypothetical statement and we are not stating that B has in fact ocurred.

Independence

Two events A and B are independent iff

$$P(A \cap B) = P(AB) = P(A)P(B)$$

Bayes Theorem

As a direct consequence of the conditional probability theorem we obtain one of the most useful theorems in probability, namely Bayes theorem.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$$

If Ω is partitioned into n independent events $(A_1, A_2, ..., A_n)$, then

$$P(A_k|B) = \frac{P(B|A_k)P(A_k)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)} \quad k \subset 1, 2, ..., n$$

One of many important applications of Bayes theorem is that it allows the computation of an inverse problem based on the direct problem. Consider as an example computing the probability of a given model(m) given experimental data(d), then

$$P(m|d) = \frac{P(d|m)P(m)}{P(d)}$$

The term $P(d|m)$ is known as the likelihood function and in this context can be interpreted as the probability of occurrence of the data given the model, *P(m)* is the prior and it reflects our belief (or confidence) in the model prior to the experiment and *P(d)* is the evidence of the probability of the data occurring.