CE-370 Uncertainty and Risk in Engineering Systems

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SUMMARY OF LECTURE 2 - COMBINATORIAL METHODS TO COMPUTE PROBABILITIES

In experiments with finite sample space (Ω) and in which all events are equiprobable, the computation of probability of events can be reduced to computing the ratio between the number of outcomes of interest and the total number of possible outcomes. Let's begin by considering the following basic result from combinatorial analysis:

R.1. The number of distinct ordered k-tuples $(x_1, x_2, \ldots, x_k)$ with components x_i from a set with n_i distinct elements is equal to $n_1, n_2, \ldots, n_k$.

Some typical applications of the previous result in the computation of probabilities are:

- *Sample with replacement and with ordering*: This corresponds to choosing k objects from a set A that has n distinct objects, with replacement. More explicitly, we select an object from set A and after identifying it and noting the order in which it came out we place it back before making our next choice.This experiment will produce a vector of size k

  $$\{x_1, x_2, \ldots, x_k\}\; x_i \in A$$

  Using (R.1) and noting that the number of possible selections for each x_i is always n, we conclude that the total number of possible outcomes $= n^k$.

  EXAMPLE: A slot machine has three dials, each has the integers from zero to nine. What is the probability on a given spin the number that results is prime? Answer: $n = 10$ and $k = 3$, therefore the total number of possible outcomes is $10^3 = 1000$ (obvious!, these are all the integers between 0 and 999), also there are 168 prime numbers between 0 and 999. Therefore the probability of obtaining a prime is 0.168. Would you bet 5-to-1 if this was a game?

- *Sample without replacement and with ordering*: This is the case where we choose k objects without replacement from a set A with n distinct objects and we record the order in which the selections are made. In this case the total number of distinct k-tuples $= n(n-1)\ldots(n-k+1)$

  EXAMPLE: Our classroom has 20 desks and we have 10 students. How many possible sitting arrangements are there? Answer: (20)(19)...(12)(11) = 670,442,572,800. So, with so many choices, why do you sit in the same place all the time?!

- *Sampling without replacement and without order*: This corresponds to selecting k objects from a set A with n distinct objects without replacement and we record the result without regards for the order in which the selections are made. It can be shown that the total number of combinations is given by the binomial coefficient

  $$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

  EXAMPLE: A certain lot from a production line has 100 units and it is known that 5 are defective. What are the chances that all 5 defective units are contained in a sub-lot of 20 items picked at random? Answer: first we must calculate all possible ways to pick 20 items out of 100. This is given by

  $$N_1 = \binom{100}{20} = \frac{100!}{20!(80)!}$$

  Then we must compute all possible ways in which one can select 5 defective items out of 5 defective in the batch *and* ($\cap$) 15 non-defective items out of a total of 95 non-defective items in the batch, that is,

  $$N_2 = \binom{5}{5}\binom{95}{15} = \frac{5!}{5!(0)!}\frac{95!}{15!(80)!}$$

  The answer is given by the ratio

  $$p = \frac{N_2}{N_1} = \frac{95!20!}{15!100!} = 0.00020593$$

  This same answer could have been obtained by computing the complementary event of sequentially selecting 80 non-defective items out of an initial 100 items where 5 are defective and 95 are non-defective. In this approach the answer is given by

  $$p = \frac{(95)(94)\cdots(16)}{(100)(99)\cdots(21)} = 0.00020593$$

  Note that the sample space is reduced every time one makes a selection of a non-defective item.

- *Sampling with replacement and without ordering*: This corresponds to the case where we select k objects from a set A containing n distinct objects with replacement and we record the result without regards to order (Note that in this case k could be larger than n). It can be shown that the number of combinations is given by

  $$\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}$$

  EXAMPLE: A bag contains five balls numbered from 1 to 5. If one selects three balls at random and after each selection the ball is identified and put back into the bag, how many possible combinations can one get if the order of selection is not important. Answer:

  $$\binom{5+3-1}{3} = \frac{(5+3-1)!}{3!(5-1)!} = 35$$