Summary of Fundamental Results and Definitions - Structural Dynamics

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Abstract

This document presents the most fundamental results and definitions for the course CE-272 Structural Dynamics. This document may grow as the class progresses, however it does not contain all information relevant to the class.

1 Definitions

Structural dynamics is concerned with predicting the time history response of structures subjected to time varying loads and(or) initial conditions.
Degree of Freedom: The number of coordinates necessary in order to describe the response of a system.
Single Degree of Freedom System (SDoF): A system whose complete motion can be described by the motion of a single coordinate.
Mass: The amount of matter contained in a body. It can also be interpreted as the force necessary in order to generate a unit acceleration.
Stiffness: The force necessary in order to generate a unit displacement.
Equilibrium position: A position such that all forces are in equilibrium in the absence of motion.
Free Vibration: The vibration that occurs when a system is imparted initial conditions in the absence of external loads or support motion.

2 Equation of Motion

The typical equation of motion of a linear SDoF is written as

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = P(t) \tag{1}$$

where m is the mass, k is the stiffness and c is the viscous damping constant. The initial conditions consist in $x(t=0) = x_o$ and $\dot{x}(t=0) = v_o$. By defining

$$\omega_n = \sqrt{\frac{k}{m}} \tag{2}$$

and

$$c = 2m\omega_n\xi \tag{3}$$

with ξ representing the fraction of critical damping, the equation of motion can be re-written as

$$\ddot{x}(t) + 2\omega_n \xi \dot{x}(t) + \omega_n^2 x(t) = \frac{P(t)}{m} \tag{4}$$

If the system is undamped, then the equation of motion simplifies to

$$\ddot{x}(t) + \omega_n^2 x(t) = \frac{P(t)}{m} \tag{5}$$

2.1 Base Motion Excitation

If a viscously damped SDoF oscillator is excited by support (or base) motion insted of a direct applied force, then the equation of motion becomes

$$\ddot{x}(t) + 2\omega_n \xi \dot{x}(t) + \omega_n^2 x(t) = -\ddot{x}_B(t) \tag{6}$$

where $\ddot{x}_B(t)$ is the acceleration time history of the base motion and $x(t)$ is the displacement of the mass relative to the base motion $x_B(t)$.

3 Free Vibration Response Formulas

3.1 Undamped Response

The free vibration response of an undamped SDoF system to initial displacement (x_o) and initial velocity (v_o) is given by

$$x(t) = x_o \cos \omega_n t + \frac{v_o}{\omega_n} \sin \omega_n t \tag{7}$$

3.2 Damped Response

The free vibration response of a viscously damped SDoF system to initial displacement (x_o) and initial velocity (v_o) is given by

$$x(t) = e^{-\omega_n \xi t} \left(x_o \cos \omega_d t + \frac{v_o + \xi \omega_n x_o}{\omega_d} \sin \omega_d t \right) \tag{8}$$

where

$$\omega_d = \omega_n \sqrt{1 - \xi^2} \tag{9}$$

The damping ratio $0 < \xi < 1$.

3.2.1 Log decrement ratio

It is possible to estimate the damping ratio ξ from examining the logarithm of the following ratio of free vibration responses

$$\delta = \frac{1}{2\pi} \log \frac{x(t + T_d)}{x(t)} \tag{10}$$

From where the damping ratio can be estimated as

$$\xi = \frac{\delta}{\sqrt{1+\delta^2}} \tag{11}$$

The following estimation can be performed for any number of subsequent number of cycles and the average value of ξ can be obtained.

4 Harmonic Load Response Formulas

4.1 Undamped Response

The response of an undamped SDoF to a harmonic load $P(t) = P_o \sin \Omega t$ and to arbitrary initial conditions x_o and v_o is given by

$$x(t) = Z_u \sin \Omega t + x_o \cos \omega_n t + \frac{v_o - Z_u \Omega}{\omega_n} \sin \omega_n t \tag{12}$$

where Z_u is given by

$$Z_u = \frac{P_o}{k} \frac{1}{1 - \left(\frac{\Omega}{\omega_n}\right)^2} \tag{13}$$

The portion of the response given by $x(t) = Z_u \sin \Omega t$ is usually denoted as the "steady-state response".

4.2 Damped Response

The response of a damped SDoF to a harmonic load $P(t) = P_o \sin \Omega t$ and to arbitrary initial conditions x_o and v_o is given by

$$x(t) = Z_d \sin(\Omega t + \phi) + e^{-\omega_n \xi t} (E \sin \omega_d t + F \cos \omega_d t) \tag{14}$$

where Z_d is given by

$$Z_d = \frac{P_o}{k} \frac{1}{\sqrt{\left(1 - \left(\frac{\Omega}{\omega_n}\right)^2\right)^2 + \left(2\xi \frac{\Omega}{\omega_n}\right)^2}} \tag{15}$$

and

$$\phi = atan \left(\frac{-2\xi \frac{\Omega}{\omega_n}}{1 - \left(\frac{\Omega}{\omega_n}\right)^2} \right) \tag{16}$$

The coefficients E and F are given by

$$E = \frac{\xi \omega_n x_o + v_o - Z_d (\xi \omega_n sin\phi + \Omega cos\phi)}{\omega_d} \tag{17}$$

$$F = x_o - Z_d sin(\phi) \tag{18}$$

The portion of the response given by $x(t) = Z_d \sin(\Omega t + \phi)$ is usually denoted as the "steady-state response".

5 Periodic Loading and Fourier Series

A periodic function is such that $f(t) = f(t+T) \quad \forall t$ where $T = \frac{2\pi}{\Omega}$ is the period of the function. Every periodic function $f(t)$ can be written as a Fourier series such that

$$f(t) = a_o + \sum_{n=1}^{\infty} a_n \cos n\Omega t + \sum_{n=1}^{\infty} b_n \sin n\Omega t \tag{19}$$

where the coefficients are given by

$$a_o = \frac{1}{T}\int_0^T f(t)dt \tag{20}$$

$$a_n = \frac{2}{T}\int_0^T \cos n\Omega t \cdot f(t)dt \tag{21}$$

$$b_n = \frac{2}{T}\int_0^T \sin n\Omega t \cdot f(t)dt \tag{22}$$

Things to keep in mind:

- A function $f(t)$ is even if $f(t) = f(-t)$. If a function is even then $b_n = 0 \forall n$.
- A function $f(t)$ is odd if $f(t) = -f(-t)$. If a function is odd then $a_n = 0 \forall n$.
- The term a_o represents the average value of the function $f(t)$.
- At a discontinuity, the Fourier series converges to the average value at the discontinuity.
- In case of discontinuities, watch out for the Gibbs phenomenon!

5.1 Response to Periodic Loading

The response of a SDoF to periodic loading $P(t)$ can be found by decomposing the load into its Fourier series and using superposition

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = P(t) = a_o + \sum_{n=1}^{\infty} a_n \cos n\Omega t + \sum_{n=1}^{\infty} b_n \sin n\Omega t \tag{23}$$

The steady-state response to the periodic loading is obtained by superimposing the steady-state response of the SDoF system to each individual wave in the Fourier series, i.e.

$$m\ddot{x}_n(t) + c\dot{x}_n(t) + kx_n(t) = a_n \cos n\Omega t + b_n \sin n\Omega t \tag{24}$$

$$x(t) = \frac{a_o}{k} + \sum_{n=1}^{\infty} x_n(t) \tag{25}$$

6 Response to Arbitrary Loading

The response of a SDoF to an arbitrary loading can be computed for any time of interest t using the following convolution integral (also known as Duhamel's integral)

$$x(t) = \int_{-\infty}^{t} h(t-\tau)P(\tau)d\tau = \int_{-\infty}^{t} h(\tau)P(t-\tau)d\tau \tag{26}$$

If $P(t) = 0 \quad \forall t \leq 0$, then the integral simplifies to

$$x(t) = \int_{0}^{t} h(t-\tau)P(\tau)d\tau = \int_{0}^{t} h(\tau)P(t-\tau)d\tau \tag{27}$$

6.0.1 Response Spectrum

The response spectrum of a load time history $P(t)$ is defined as

$$\max r(t) \quad \forall t \quad s.t. \quad r(t) = f(x(t)) \tag{28}$$

where $f(x(t))$ is a function of $x(t)$ such as displacement, velocity, reaction force, etc.

7 Laplace Domain Analysis

The Laplace transform $\mathcal{X}(t)$ of a function $x(t)$ is defined as

$$\mathcal{X}(s) = \mathcal{L}(x(t)) = \int_{0}^{\infty} x(t)e^{-st}dt \tag{29}$$

It can be shown that

$$\mathcal{L}(\dot{x}(t)) = s\mathcal{L}(x(t)) = s\mathcal{X}(s) - x_o \tag{30}$$

Similarly

$$\mathcal{L}(\ddot{x}(t)) = s^2\mathcal{L}(x(t)) + sx_o + v_o = s^2\mathcal{X}(s) - sx_o - v_o \tag{31}$$

Applying these results to the equation of motion of linear SDOF systems we obtain

$$ms^2\mathcal{X}(s) + cs\mathcal{X}(s) + k\mathcal{X}(s) = \mathcal{P}(s) + x_o\,(1+s) + v_o \tag{32}$$

The solution, in the Laplace domain is given by

$$\mathcal{X}(s) = \frac{\mathcal{P}(s) + x_o\,(1+s) + v_o}{ms^2 + cs + k} \tag{33}$$

In the special case of zero initial conditions

$$\mathcal{X}(s) = \mathcal{P}(s)\frac{1}{ms^2 + cs + k} = \mathcal{P}(s)\mathcal{H}(s) \tag{34}$$

This happily turns out to be an algebraic equation! One of the advantages of working in the Laplace domain is that one can turn a linear differential

equation into an algebraic equation. The function $\mathcal{H}(s)$ is known in control literature as the Transfer function.
The transfer function can be re-written as

$$\mathcal{H}(s) = \frac{1}{ms^2 + cs + k} = \frac{1}{m(s^2 + 2\omega_n \xi s + \omega_n^2)} = \frac{1}{m(s - s_1)(s - s_2)} \tag{35}$$

where

$$s_{1,2} = -\omega_n \xi \pm i\omega_d \tag{36}$$

The values $s_{1,2}$ are known as the poles of the transfer function. It is trivial to show that $\forall 0 < \xi < 1$

$$|s_{1,2}| = \omega_n \tag{37}$$

i.e. the poles of the transfer function lie in a circle of radius ω_n.
In order to return to the time domain, one must apply an inverse Laplace transform to $\mathcal{X}(s)$. This involves using the Bromwich formula

$$x(t) = \lim_{R \to \infty} \int_{a-iR}^{a+iR} \frac{1}{2\pi i} \mathcal{X}(s) e^{st} ds \tag{38}$$

where a is taken to the right of all the singularities of $\mathcal{X}(s)$. There are also many available tables of Laplace transform pairs, which can be very useful and save time.

8 Fourier Domain Analysis

The Fourier transform is the Laplace transform evaluated along the imaginary axis, so $s = i\omega$.

$$\mathcal{X}(i\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt \tag{39}$$

The inverse Fourier transform is given by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{X}(i\omega) e^{i\omega t} d\omega \tag{40}$$

The relationship between force and displacement in the Fourier domain (also commonly referred to as frequency domain) is given by

$$\mathcal{X}(\omega) = \mathcal{P}(\omega) \frac{1}{-m\omega^2 + i\omega c + k} \tag{41}$$

A useful result is

$$\mid \mathcal{X}(\omega) \mid = \mid \mathcal{P}(\omega) \mid \frac{1/m}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\omega_n^2 \omega^2 \xi^2}} \tag{42}$$

9 Multi-Degree of Freedom System

In a structural system with n interconnected masses it is necessary to account for the forces being transmitted through the connections. The equation of motion of every mass will include coupling terms that involve the motion of one or more of the remaining masses. In the case of linear viscous damping, the m equation of motions can be written in matrix form as

$$\mathbf{M}\ddot{x} + \mathbf{C}\dot{x} + \mathbf{K}x = f(t) \tag{43}$$

The three resulting matrices are typically denoted as: **M**, the mass matrix, **C** damping matrix and **K** is the stiffness matrix. The time varying vector $\mathbf{x(t)}$ is the displacement of the **m** masses at time **t**. The right hand side $f(t)$ is the force vector at time t applied at all masses.

10 Modal Analysis

The objective of modal analysis is to decouple the equations of motion. We wish to transform the set of m interdependent equations to m independent equations that can be solved one-at-a-time. To begin, consider the undamped equation of motion

$$\mathbf{M}\ddot{x} + \mathbf{K}x = 0 \tag{44}$$

Assume a solution of the form

$$x(t) = A\phi \sin \omega t \tag{45}$$

and second derivative given by

$$\ddot{x}(t) = -A\phi\omega^2 \sin \omega t \tag{46}$$

where A is a scalar, ϕ is a vector, and ω is a circular frequency. Substituting this *ansatz* into the equation of motion we obtain

$$-\mathbf{M}\phi\omega^2 \sin \omega t + \mathbf{K}A\phi \sin \omega t = 0 \tag{47}$$

Eliminating common terms we get

$$\mathbf{K}\phi = \mathbf{M}\phi\lambda \tag{48}$$

This can be re-written in two common equivalent forms, such as

$$(\mathbf{K} - \mathbf{M}\lambda)\,\phi = 0 \tag{49}$$

$$\mathbf{M}^{-1}\mathbf{K}\phi = \phi\lambda \tag{50}$$

These equations mean that if you can find any ϕ and λ that satisfy these equations, then the *ansatz* is correct. Eq.51 can be useful when trying to determine an analytical expression to find λ.

$$(\mathbf{K} - \mathbf{M}\lambda)\,\phi = 0 \Rightarrow det\,(\mathbf{K} - \mathbf{M}\lambda) = 0 \tag{51}$$

This results in a polynomial of degree n with n roots. Each one of these roots is a solution with a corresponding vector ϕ. The corresponding ϕ is a vector in the null-space of the matrix $(\mathbf{K} - \mathbf{M}\lambda)$. The scalar λ is known as *eigenvalue* and the corresponding vector ϕ is known as an eigenvector.

10.1 Orthogonality Property

It can be shown that eigenvectors of the undamped system satisfy the following orthogonality property for $i \neq j$

$$\phi_i^T \mathbf{M} \phi_j = 0 \tag{52}$$

Similarly

$$\phi_i^T \mathbf{K} \phi_j = 0 \tag{53}$$

10.2 Mass-normalized modes

If ϕ is a mode shape (arbitrarily scaled), then $\alpha\phi$ is also a mode shape, where α is a scalar. Thus it is possible to find scalars α_i for each mode i such that

$$\psi_i^T \mathbf{M} \psi_i = 1 \tag{54}$$

where $\psi = \alpha\phi$. The scaled modes ψ are known as mass-normalized modes. The corresponding normalizing scalars for each mode are given by

$$\alpha_i = \frac{1}{\sqrt{\phi_i^T \mathbf{M} \phi_i}} \tag{55}$$

For mass-normalized modes, the following equality holds

$$\psi_i^T \mathbf{K} \psi_i = \lambda_i \tag{56}$$

10.3 Classical Damping

Any viscous damping matrix that satisfies the following orthogonality property is known as classical damping

$$\psi_i^T \mathbf{C} \psi_j = 0 \tag{57}$$

Simple examples of Classical Damping matrices

$$\mathbf{C} = a\mathbf{M} \qquad \mathbf{C} = b\mathbf{K} \qquad \mathbf{C} = a\mathbf{M} + b\mathbf{K} \tag{58}$$

10.4 Modal Analysis

For systems with classical damping

$$\mathbf{M}\ddot{x} + \mathbf{C}x + \mathbf{K}x = 0 \tag{59}$$

One can define the following linear transformation of the displacement vector

$$x(t) = \Psi z(t) \tag{60}$$

where

$$\Psi = [\psi_1 \quad \psi_2 \quad ... \quad \psi_n] \tag{61}$$

then

$$\mathbf{M}\Psi\ddot{z} + \mathbf{C}\Psi\dot{z} + \mathbf{K}\Psi z = f(t) \tag{62}$$

premultiplying by Ψ^T we obtain

$$\Psi^T\mathbf{M}\Psi\ddot{z} + \Psi^T\mathbf{C}\Psi\dot{z} + \Psi^T\mathbf{K}\Psi z = \Psi^T f(t) \tag{63}$$

Using the aforementioned orthogonality properties

$$\ddot{z} + \Xi\dot{z} + \Lambda z = \Psi^T f(t) \tag{64}$$

these are independent set of equations and hence can be solved individually for each mode i.

$$\ddot{z}_i + 2\omega_i\xi_i\dot{z}_i + \omega_i^2 z_i = \psi_i^T f(t) \tag{65}$$

The dynamic response in physical coordinates can be found by simply applyimng the linear transformation

$$x(t) = \Psi z(t) = \sum_i \psi_i z_i(t) \tag{66}$$

11 Numerical Methods to Compute Dynamic Response

Below we present to popular and effective methods to approximate the equation of motion.

11.1 Central Difference Method

The CD method approximates the velocity and acceleration as follows

$$\dot{x}(t) = \frac{x(t+\Delta t) - x(t-\Delta t)}{2\Delta t} \tag{67}$$

$$\ddot{x}(t) = \frac{\dot{x}(t+\Delta t) - \dot{x}(t-\Delta t)}{2\Delta t} = \frac{x(t+2\Delta t) - 2x(t) + x(t-2\Delta t)}{4\Delta t^2} \tag{68}$$

Substituting these approximations into the equation of motion we obtain the following recursive expression for $x(t)$

11.2 Newmark's Constant Acceleration Method

The NCA method assumes the acceleration to be constant within a time step and proceeds to estimate the velocity and displacement based on that. Therefore

$$\ddot{x}(t+\tau) = \ddot{x}(t) \quad \forall \quad 0 < \tau < \Delta t \tag{69}$$

$$\dot{x}(t+\tau) = \ddot{x}(t)\tau + \dot{x}(t) \quad \forall \quad 0 < \tau < \Delta t \tag{70}$$

$$x(t+\tau) = \frac{1}{2}\ddot{x}(t)\tau^2 + \dot{x}(t)\tau + x(t) \quad \forall \quad 0 < \tau < \Delta t \tag{71}$$

These approximations are then inserted into the equation of motion to obtain