where the incomplete Beta function is given by $\beta_S(q,r)=\int_0^S y^{q-1}(1-y)^{r-1}dy$.

The first two moments and the coefficient of skewness of the Beta distribution are:

$$E(X)=\mu_X=a+\frac{q(b-a)}{q+r} \tag{A.69}$$

$$\mathrm{var}(X)=\sigma_X^2=\frac{qr(b-a)^2}{(q+r)^2(q+r+1)} \tag{A.70}$$

$$\gamma_1=\frac{2(r-q)}{(q+r)(q+r+2)\,\sigma_X} \quad \text{(skewness coefficient)} \tag{A.71}$$

The *incomplete Beta function ratio* $\beta_S(q,r)/\beta(q,r)$ has been tabulated [Pearson and Johnson, 1968]. If q,r are both integral, $BT(0,1,q,r)$ is binomially distributed such that

$$f_S(s)=(q+r-1)p_X(x) \tag{A.72}$$

where $p_X(x)$ is binomially distributed as $B(q+r-2,s)$ with $x=q-1$.

A special case of the general Beta distribution is the *rectangular* or *uniform* distribution $BT(a,b,1,1)=R(a,b)$ with probability density function and cumulative distribution function given by:

$$\begin{aligned} f_X(x)&=\frac{1}{b-a} && a<x<b \\ &=0 && \text{elsewhere} \end{aligned} \tag{A.73}$$

$$\begin{aligned} F_X(x)&=\frac{x-a}{b-a} && a<x<b \\ &=0 && x\le a \\ &=1 && x\ge b \end{aligned} \tag{A.74}$$

with moments

$$\mu_X=\frac{(a+b)}{2} \qquad \sigma_X^2=\frac{(b-a)^2}{12} \tag{A.75}$$

A.5.11 Extreme value distribution type I *EV-I*(μ, α)

This is the limiting (asymptotic) distribution of the largest (smallest) of n random variables X_i as $n\to\infty$. The distribution of the X_i must be of the form $F_X(x)=1-\exp[-g(x)]$ or $f_X(x)=\exp[-g(x)]$ with $\mathrm{d}g/\mathrm{d}x>0$. The normal, gamma and exponential distributions are of this type. If Y is the *largest* of many independent X_i then its probability density and cumulative distribution functions are, asymptotically, given by the following expressions [Gumbel, 1958]:

$$f_Y(y) = \alpha \exp[-\alpha(y-u) - e^{-\alpha(y-u)}] \qquad -\infty < y < \infty \qquad \text{(A.76)}$$

$$F_Y(y) = \exp[-e^{-\alpha(y-u)}] \qquad -\infty < y < \infty \qquad \text{(A.77)}$$

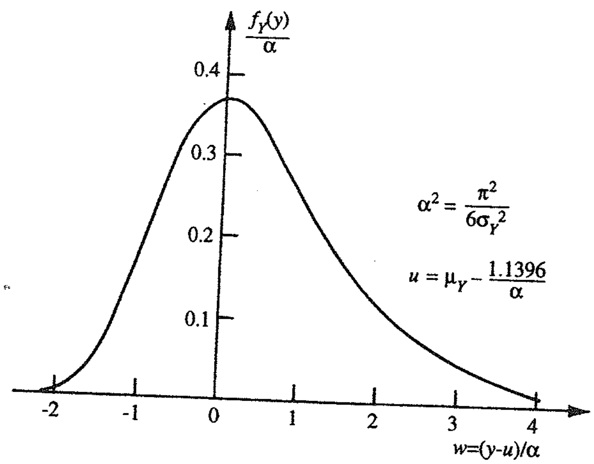


Figure A.5 Extreme value distribution type I (Gumbel)

The parameters are the mode u of the distribution and α which is the measure of the dispersion of the distribution. α^{-1} is sometimes known as the 'slope' of the distribution (obtained when plotting the distribution on so-called 'Gumbel' paper). Both u and α may be obtained, via the moments, from curve fitting to observed data. The moments are:

$$E(Y) = \mu_Y = u + \gamma / \alpha \qquad \text{(A.78)}$$

$$\text{var}(Y) = \sigma_Y^2 = \frac{\pi^2}{6\alpha^2} \qquad \text{(A.79)}$$

$$\gamma_1 = 1.1396 \qquad \text{(skewness)} \qquad \text{(A.80)}$$

where $\gamma = 0.577\ 215\ 664\ 9...$ is Euler's constant and the skewness is seen as independent of u and α. The following points might be noted in applications using this distribution:

(1) In practice, the X_i of the underlying population need not be completely independent or completely identical [Gumbel, 1958]. Also, it may be difficult to determine the appropriate underlying distribution of the X_i, and convergence to the asymptotic distribution may be slow. Nevertheless extreme value distributions are useful for fitting to experimental data even where the underlying mechanisms are not fully understood.

(2) The distribution usually is tabulated in terms of a reduced variate $W = (Y-u)\alpha$ for which $u = 0, \alpha = 1$ and $F_W(w) = \exp[-e^{-w}]$ [National Bureau of Standards, 1953] The

probability density function and the cumulative distribution function in terms of Y are recovered from

$$f_Y(y) = \alpha f_W[(y-u)\alpha] \tag{A.81}$$

$$F_Y(y) = F_W[(y-u)\alpha] \tag{A.82}$$

(3) This distribution is also termed the 'double exponential', 'Gumbel' or 'Fisher-Tippett Type I' distribution.

The complementary result is as follows. The probability density function and the cumulative distribution function for the *smallest* value Y^S of many independent X_i are given by, respectively:

$$f_{Y^S}(y^S) = \alpha \exp[\alpha(y^S - u) - e^{\alpha(y^S-u)}] \quad -\infty < y^S < \infty \tag{A.83}$$

$$F_{Y^S}(y^S) = 1 - \exp[-e^{\alpha(y^S-u)}] \quad -\infty < y^S < \infty \tag{A.84}$$

with moments

$$\mu_{Y^S} = u - \gamma / \alpha \tag{A.85}$$

$$\sigma^2_{Y^S} = \frac{\pi^2}{6\alpha^2} \tag{A.86}$$

$$\gamma_1 = -1.1396 \tag{A.87}$$

The tabulated results for the reduced variable W described above can be applied since the distribution for Y^S is related to that for W by

$$f_{Y^S}(y^S) = \alpha f_W[-(y^S - u)\alpha] \tag{A.88}$$

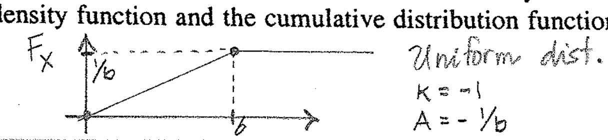
$$F_{Y^S}(y^S) = 1 - F_W[-(y^S - u)\alpha] \tag{A.89}$$

The extreme value distribution for the minimum value has less practical application than that for the maximum value; the Weibull distribution (extreme value distribution type III) is more comonly used for smallest values.

A.5.12 Extreme value distribution type II *EV-II(u,k)*

$f_X(x) = AKx^{-(k+1)}$

This is the limiting distribution of the largest of n random variables X_i as $n \to \infty$. The distribution of the X_i must be of the form $F_X(x) = 1 - Ax^{-k}$, $x \geq 0$, A = constant [Gumbel, 1958]. Typical of this form is the Pareto distribution and the Cauchy distribution for $x \geq 0$. The probability density function and the cumulative distribution function are, respectively:



$$f_Y(y) = \frac{k}{y}\left(\frac{u}{y}\right)^k e^{-(u/y)^k} \qquad y \geq 0 \qquad \text{(A.90)}$$

$$F_Y(y) = e^{-(u/y)^k} \qquad y \geq 0 \qquad \text{(A.91)}$$

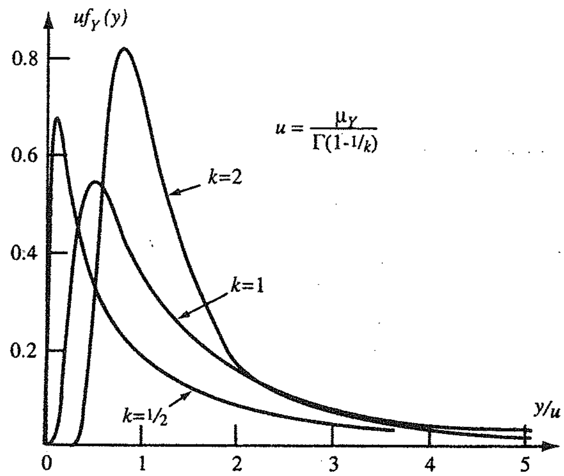


Figure A.6 Extreme value distribution type II (Frechet).

The parameters are the characteristic value u of the distribution (median > u > mode; median mode for $k > 4$) and k which is a dimensionless inverse measure of the dispersion of the distribution. The first two moments are:

$$E(Y) = \mu_Y = u\Gamma(1 - 1/k) \qquad k > 1 \qquad \text{(A.92)}$$

$$\text{var}(Y) = \sigma_Y^2 = u^2\left[\Gamma(1 - 2/k) - \Gamma^2(1 - 1/k)\right] \qquad k > 2 \qquad \text{(A.93)}$$

so that $$V_Y^2 = \frac{\mu_Y^2}{\sigma_Y^2} = \frac{\Gamma(1-2/k)}{\Gamma^2(1-2/k)} - 1 \qquad \text{(A.94)}$$

Moments of order $l \geq k$ do not exist; this complicates the estimation of u and k.

The following points should be noted in applications using this distribution.

(1 If it is known that $k > 2$, equation (A.94) may be used to evaluate k, and then u may be evaluated from (A.92).

(2) The type II distribution for Y. EV-II(u,k), may be transformed to the type I for Z, EV-I(u, α), by letting $Z = \ln Y$. Then

$$f_Y(y) = \frac{1}{y} f_Z(\ln y) \qquad \text{(A.95)}$$

$$F_Y(y) = F_Z(\ln y) \tag{A.96}$$

$$\alpha = k \tag{A.97}$$

Hence, in terms of the reduced variable W, which is tabulated (see Section A.5.11),

$$f_Y(y) = \frac{k}{y} f_W[(\ln y - \ln u)k] \tag{A.98}$$

$$F_Y(y) = F_W[(\ln y - \ln u)k] \tag{A.99}$$

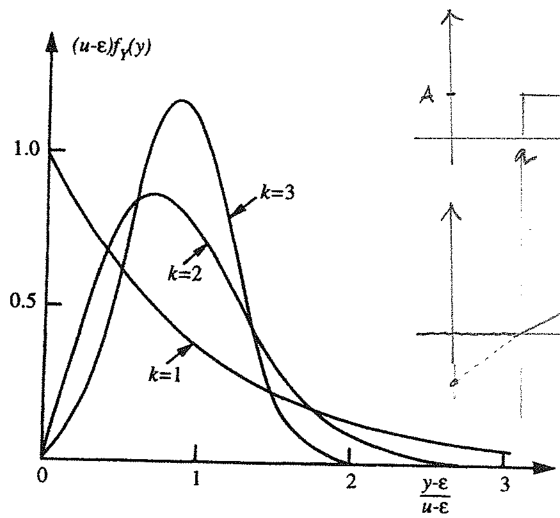
(3) The above properties hold for $y \geq 0$. A more general result, for $y \geq \varepsilon$, $\varepsilon \neq 0$, can be obtained by linear transformation by writing $u - \varepsilon$ for u and $y - \varepsilon$ for y.
(4) This distribution is sometimes known as the 'Frechet' distribution.
(5) The distribution for the smallest extreme value is of no practical interest.
(6) The underlying distributions X_i for the type II distribution typically have longer tails ($x \geq 0$) than those for the type I distribution.

A.5.13 Extreme value distribution type III *EV-III(ε,u,k)*

This represents the (asymptotic) distribution of the largest (smallest) value of n random variables X_i as $n \to \infty$, with X_i limited *in the tail of interest* to some maximum (minimum) value w (or ε), and X_i having a distribution of general form

$$F_X(x) = 1 - A(w - x)^k \qquad x \leq w, k > 0, A = \text{constant}$$

The rectangular ($k = 1$), triangular ($k = 2$) and the Gamma distribution ($\varepsilon = 0$) are of this form. The probability density function and the cumulative distribution function for the largest value Y^L of many independent X_i are given, respectively, by [Gumbel, 1958]:



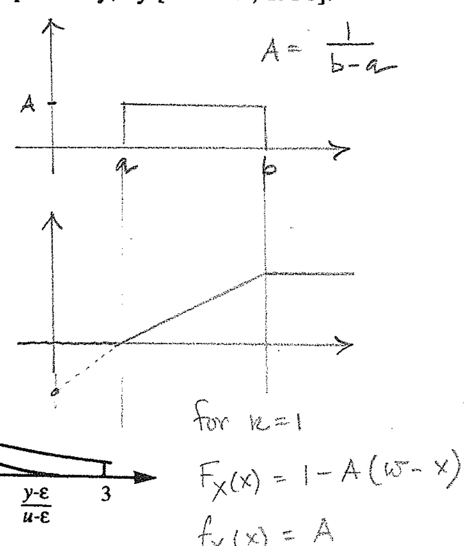


Figure A.7 Extreme value distribution type III (Weibull).

$$f_{Y^L}(y^L) = \frac{k}{w-u}\left(\frac{w-y^L}{w-u}\right)^{k-1} F_{Y^L}(y^L) \qquad y^L \le w \qquad \text{(A.100)}$$

$$F_{Y^L}(y^L) = \exp\left[-\left(\frac{w-y^L}{w-u}\right)^{k}\right] \qquad y^L \le w \qquad \text{(A.101)}$$

More useful is the distribution of the *smallest* value Y of many independent X_i. The relevant cumulative distribution and the probability density functions are [Gumbel, 1958]:

$$F_Y(y) = P(Y \le y) = 1 - P_Y(y) \qquad y \ge \varepsilon \qquad \text{(A.102)}$$

where

$$P_Y(y) = \exp\left[-\left(\frac{y-\varepsilon}{u-\varepsilon}\right)^{k}\right] \qquad y \ge \varepsilon \qquad \text{(A.103)}$$

which equals the probability of a value Y larger than y, i.e. $P(Y > y)$. Also,

$$f_Y(y) = \frac{\mathrm{d}F_Y(y)}{\mathrm{d}y} = \frac{k}{u-\varepsilon}\left(\frac{y-\varepsilon}{u-\varepsilon}\right)^{k-1} P_Y(y) \qquad y \ge \varepsilon \qquad \text{(A.104)}$$

Th parameters are the minimum value ε of X_i (and hence Y), the characteristic value u of the distribution (which converges to μ_Y as $k \to \infty$) and the 'scale parameter' $1/k$ (usually $k > 1$). The moments are

$$E(y) = \mu_Y = \varepsilon + (u-\varepsilon)\Gamma(1+1/k) \qquad \text{(A.105)}$$

$$\mathrm{var}(y) = \sigma_Y^2 = (u-\varepsilon)^2[\Gamma(1+2/k) - \Gamma^2(1+1/k)] \qquad \text{(A.106)}$$

The following points should be noted in application of this distribution:

(1) Estimation of the parameters ε, u and k generally is not straightforward. If the underlying distribution in is known, k is known and ε and u can be estimated from the estimates for μ_Y and σ_Y^2. Otherwise, k may be estimated from sample skewness or u may be estimated from order statistics [Gumbel, 1958]. If the lower limit ε is known, or is zero, then u and k can be evaluated from equations (A.105) and (A.106) by writing y for $y-\varepsilon$ and hence

$$\mu_Y = u\Gamma(1+1/k)$$

$$\sigma_Y^2 = u^2[\Gamma(1+2/k) - \Gamma^2(1+1/k)]$$

and

$$1 + V_Y^2 = \frac{\Gamma(1+2/k)}{\Gamma^2(1+1/k)} \quad \text{or} \quad k \approx V_Y^{-1.09}$$

all of which can be estimated from sample data [Gumbel, 1958]. However, the procedure may be cumbersome [see also Mann *et al.*, 1974].

(2) The distribution $F_Y(y)$ is pseudo-symmetric for 3.2< k<3.7.

(3) If Y is *EV-III* (ε,u,k) for smallest values, then $Z=\ln(Y-\varepsilon)$ is *EV-I* $[\ln(y-\varepsilon),k]$ for smallest values. This enables the third extreme value distribution to be evaluated using the tables for *EV-I* (largest) in terms of the reduced variate W:

$$F_Y(y)=1-F_W\{-k[\ln(y-\varepsilon)-\ln(u-\varepsilon)]\} \qquad y\geq\varepsilon \qquad \text{(A.107)}$$

$$f_Y(y)=\frac{k}{y-\varepsilon}f_W\left[-k\ln\left(\frac{y-\varepsilon}{u-\varepsilon}\right)\right] \qquad y\geq\varepsilon \qquad \text{(A.108)}$$

(4) The distribution $P_Y(y)$ is also known as the Weibull distribution.

(5) If $\varepsilon=0,\ k=2$ the distribution is also known as the Rayleigh distribution:

$$f_Y(y)=\frac{y}{\sigma_Y^2}\exp\left(-\frac{y^2}{2\sigma_Y^2}\right) \qquad \text{(A.102a)}$$

$$F_Y(y)=1-\exp\left(-\frac{y^2}{2\sigma_Y^2}\right) \qquad \text{(A.103a)}$$

A.6 JOINTLY DISTRIBUTED RANDOM VARIABLES

A.6.1 Joint probability distribution

If an event is the result of two (or more) continuous random variables, X_1 and X_2 say, the probabilities that the event occurs for given values x_1 and x_2 are described by the joint cumulative distribution function

$$\begin{aligned}F_{X_1X_2}(x_1,x_2)&=P[(X_1\leq x_1)\cap(X_2\leq x_2)]\geq 0\\ &=\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}f_{X_1X_2}(u,v)\,du\,dv\end{aligned} \qquad \text{(A.109)}$$

where $f_{X_1X_2}(x_1,x_2)\geq 0$ is the joint probability density function. Evidently, if the partial derivatives exist,

$$\begin{aligned}f_{X_1X_2}(x_1,x_2)&\equiv\lim_{\delta x_1,\delta x_2\to 0}\{P[(x_1<X_1\leq x_1+\delta x_1)\cap(x_2<X_2\leq x_2+\delta x_2)]\}\\ &=\frac{\partial^2F_{X_1X_2}(x_1,x_2)}{\partial x_1\partial x_2}\end{aligned} \qquad \text{(A.110)}$$