# CE-395 Safety and Reliability of Engineering Systems

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## THE WEIBULL DISTRIBUTION AND THE BATHTUB CURVE

### Definition

The Weibull distribution is one of most important and widely used distributions in probability, statistics and reliability of engineering systems. It was originally proposed by Waloddi Weibull [1]. In reliability analysis, the Weibull random variable has found application, among other things, to model the behavior of systems and components during their various operational phases. Specifically:

- Infant Mortality: This phase is characterized by decreasing failure rate. This occurs in systems during their early operation where there is significant uncertainty on whether the system or component was fabricated properly. In this phase, the confidence on the system performance increases as time elapses. Note that, the actual safety of the system does not increase, but our confidence in the system performance does.
- Normal Life: In this phase the system is operating under normal conditions and failure is mainly due to random phenomena. this phase should be characterized by low failure rates.
- End of Life (Wear-out): In this phase the system's failure rate increases with time due to wear-out and degradtion.

The Weibull random variable $T$ is only defined on the interval $[0 \quad \infty)$ and the probability measure of the sets $[0 \quad x]$ is given by the function

$$F_T(t) = 1 - e^{-(\lambda t)^k} \tag{1}$$

where $k$ is the shape parameter and $\lambda$ is the scale parameter. Differentiating we obtain its probability density function given by

$$f_T(t) = k\lambda \left(\lambda t\right)^{k-1} e^{-(\lambda t)^k} \quad \text{for} \quad t \geq 0 \tag{2}$$

with mean time to failure given by

$$\mathbb{E}(T) = \lambda \Gamma \left(1 + \frac{1}{k}\right) \tag{3}$$

Based on first principles of probability theory (see Lecture notes 2 and 7), we find that its reliability is given by

[1] ASME Journal of Applied Mechanics Division, 293-297, 1951

$$R(t) = 1 - F_T(t) = e^{-(\lambda t)^k} \tag{4}$$

and its failure rate function by

$$r(t) = k\lambda\,(\lambda t)^{k-1} \tag{5}$$

Therefore if

- $k < 1$ the failure rate will be decreasing with time, thus it is possible to model infant mortality of systems and(or) components.
- $k = 1$ the failure rate will be constant ($\lambda$) and this models the normal operational life of systems and(or) components.
- $k > 1$ the failure rate will be increasing and this can be used to model the wear-out phase of system and(or) components.

**BATHTUB CURVE**

The previous discussion leads to the idea of the "bathtub" curve. This curve describes the evolution of the failure rate $r(t)$ during the life of a system and(or) component. Consider as an example a system with three phases characterized by:

- $k = \frac{1}{2}$ and $\lambda = 1$ for the interval $(0 \quad 1]$ - Infant Mortality
- $k = 1$ and $\lambda = \frac{1}{2}$ for the interval $(1 \quad 8]$ - Normal Life
- $k = 3$ and $\lambda = \left(\frac{1}{384}\right)^{\frac{1}{3}}$ or the interval $(8 \quad \infty)$ - Wear-out

We can find (see below) the corresponding failure rate and perform the required integration to obtain the corresponding reliability function (See Lecture Notes 7).