CE-370 Safety and Reliability of Engineering Systems

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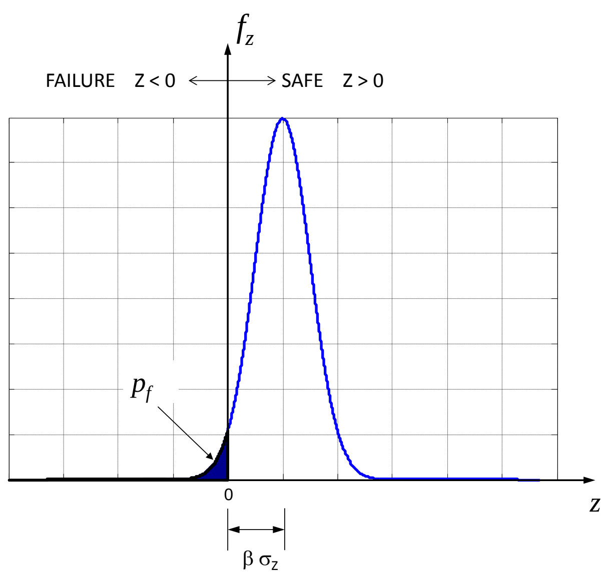
SUMMARY OF LECTURE 13 - FIRST-ORDER RELIABILITY METHOD (FORM)

In many structural reliability problems we are interested in the random variable $Z = R - Q$ where R is the resistance and Q is the loading effect. If R and Q are independent Gaussian random variables, then Z will be Gaussian with mean $\mu_Z = \mu_R - \mu_Q$ and variance $\sigma_Z^2 = \sigma_R^2 + \sigma_Q^2$. The variable Z defines the failure function, and $Z < 0$ denotes failure.



If we define the reliability index as

$$\beta = \frac{\mu_Z}{\sigma_Z} \tag{1}$$

We can show that the probability of failure p_f given by

$$p_f = \Phi(-\beta)$$

where $\Phi(\cdot)$ is the cumulative function of the standard random variable given by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-t^2}{2}} dt \qquad x > 0$$

Numerical estimates of $\Phi(\cdot)$ can be found by approximate analytical formulas and numerical methods. These results have been gathered in tables which can be found in numerous books on mathematical tables.

MULTIVARIATE CASE

In a more general case, for any linear failure function given by

$$g(X) = a_0 + a_1 X_1 + \cdots + a_n Xn \tag{2}$$

we are interested in the event $g(X) \leq 0$. In the case of uncorrelated random variables, the Hasofer-Lind transformation

$$Y_i = \frac{X_i - \mu_i}{\sigma_i} \tag{3}$$

transforms all variables to zero mean and unit variance. In the case of correlated random variables it is necessary to transform all variables

$$y = \mathbf{T}x \tag{4}$$

where

$$\mathbf{T} = \Lambda^{-\frac{1}{2}} \mathbf{L}^T \tag{5}$$

The matrices L and Λ satisfy the relationship

$$\mathbf{K}_X = \mathbf{L} \Lambda \mathbf{L}^T \tag{6}$$

where K_X is the covariance matrix of X, Λ is a diagonal matrix and due to the symmetry of K_X, the matrix L is orthogonal.

To extend the idea of reliability index to the multidimensional case we simply find the closest distance from the origin in the Y space to the failure surface defined by $g(Y) = 0$. To achieve this find the gradient of the failure surface at the point closest to the origin. We call this point the "design point".

The components of the gradient vector are given by

$$c_i = \lambda \frac{\partial g}{\partial y_i} \tag{7}$$

where *lambda* is arbitrary. The Euclidean length of the gradient vector is given by

$$l = \left(\sum_i c_i^2 \right)^{1/2} \tag{8}$$

The directional cosines of the unit gradient vector is then given by

$$\alpha_i = \frac{c_i}{l} \tag{9}$$

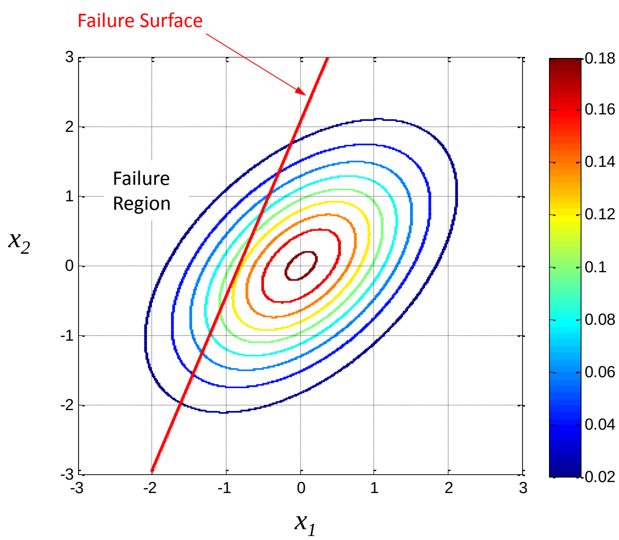
The design point can be expressed as

$$y_i^* = -\alpha_i \beta \tag{10}$$

where β is the "length" of the line joining the origin and the failure surface. The constant /*beta* is the multivariate equivalent of the reliability index.

EXAMPLE

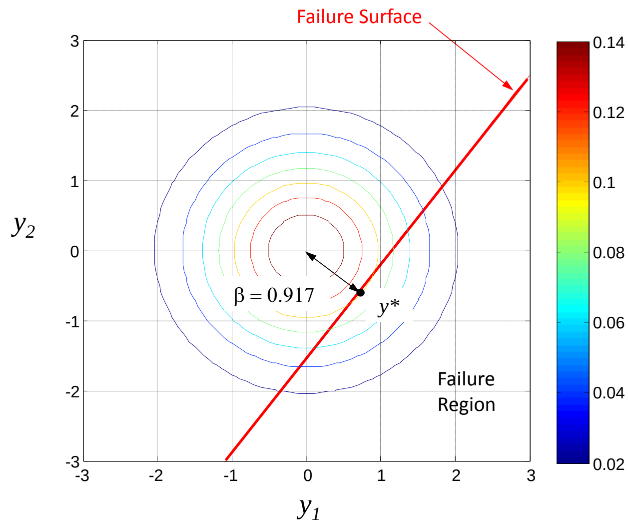
Consider a bivariate case with failure function described by $g(x) = 2 + \frac{5}{2}x_1 - x_2$. The variables are jointly Gaussian with zero means, unit variances and correlation coefficient $\rho = 0.5$.



By applying a linear transformation $y = \mathbf{T}x$ we can obtain normalized Gaussian random variables and modified failure function given by $g(y) = 2 - 1.75y_1 + 1.299y_2$. The direccional cosines of the line perpendicular to the failure surface are found by implementing eq.7-9. We obtain $c_1 = -1.75$ and $c_2 = 1.299$ which yields $\alpha = 53.41$ degrees.

$$\begin{bmatrix} 0.742 & 1 \\ -1.75 & 1.299 \end{bmatrix} \begin{bmatrix} y_1^* \\ y_2^* \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

Note that due to the transformation of the variables the failure region has been flipped.



Failure Surface
β = 0.917
y*
Failure
Region
y1
y2
0.14
0.12
0.1
0.08
0.06
0.04
0.02