

CE-370 Safety and Reliability of Engineering Systems

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SUMMARY OF LECTURE 10 - FUNCTIONS OF A RANDOM VARIABLE

In many situations one is interested in determining the cumulative distribution function (CDF) and probability density function (PDF) of a certain function of a random variable with known CDF and PDF. That is, given $F_X(x)$, $f_X(x)$ and the function

$$Y = \phi(X) \quad (1)$$

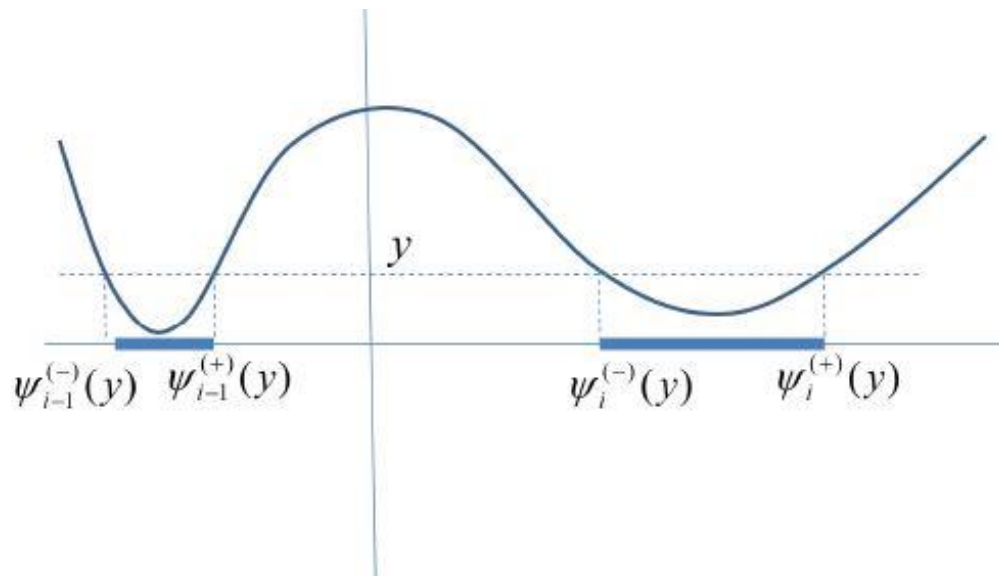
we are interested in

$$F_Y(y) = P[Y \leq y] \quad (2)$$

and its derivative

$$f_Y(y) = \frac{d}{dy} F_Y(y) \quad (3)$$

To begin, consider the figure below depicting an arbitrary function $y = \phi(x)$



As can be seen, the event $Y \leq y$ is equivalent to the event $\bigcup_i \psi_i^{(-)}(y) \leq x \leq \psi_i^{(+)}(y)$ where $\psi(\cdot) = \phi^{-1}(\cdot)$. Therefore

$$F_Y(y) = P[Y \leq y] \Leftrightarrow \sum_i P[\psi_i^{(-)} \leq x \leq \psi_i^{(+)}(y)] = \sum_i (F_X(\psi_i^{(+)}(y)) - F_X(\psi_i^{(-)}(y))) \quad (4)$$

Consequently, the probability density function (PDF) of Y is given by

$$f_Y(y) = \frac{df_Y(y)}{dy} = \sum_i \frac{d(F_X(\psi_i^{(+)}(y)) - F_X(\psi_i^{(-)}(y)))}{dy} \quad (5)$$

and using the chain rule of differentiation we obtain a general expression for $f_Y(y)$

$$f_Y(y) = \sum_i \left(\frac{d\psi_i^{(+)}(y)}{dy} f_X(\psi_i^{(+)}(y)) - \frac{d\psi_i^{(-)}(y)}{dy} f_X(\psi_i^{(-)}(y)) \right) \quad (6)$$

SPECIAL CASE 1: MONOTONICALLY INCREASING FUNCTION If $\phi(x)$ increases monotonically, the results in eq. 6 and 4 simplify to

$$F_Y(y) = F_X(\psi(y)) - F_X(-\infty) = F_X(\psi(y)) \quad (7)$$

$$f_Y(y) = \frac{d\psi(y)}{dy} f_X(\psi(y)) \quad (8)$$

SPECIAL CASE 2: MONOTONICALLY DECREASING FUNCTION If $\phi(x)$ decreases monotonically, the results in eq. 6 and 4 simplify to

$$F_Y(y) = F_X(+\infty) - F_X(\psi(y)) = -F_X(\psi(y)) \quad (9)$$

$$f_Y(y) = -\frac{d\psi(y)}{dy} f_X(\psi(y)) \quad (10)$$

Note that since the function is monotonically decreasing, the derivative is negative and thus

$$f_Y(y) = \left| \frac{d\psi(y)}{dy} \right| f_X(\psi(y)) \quad (11)$$

which is actually an expression for any monotonic function.

EXPECTED VALUE OF FUNCTIONS OF RANDOM VARIABLES

The expected value of a function $y = \phi(x)$ of a random variable X is given by

$$E[Y] = \int_{-\infty}^{+\infty} \phi(x) f_X(x) dx \neq \phi(E[X]) \quad (12)$$

This expression is valid for any arbitrary function $\phi(x)$ monotonic or not.

VARIANCE OF FUNCTIONS OF RANDOM VARIABLES

By definition

$$VAR[Y] = E[Y^2] - E[Y]^2 \quad (13)$$

where

$$E[Y^2] = \int_{-\infty}^{+\infty} \phi^2(x) f_X(x) dx \quad (14)$$