

CE-370 Safety and Reliability of Engineering Systems

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FROM BERNOULLI TO POISSON

Bernoulli Random Variable

The sample space for a Bernoulli random variable X is binary, typically it is defined by $\Omega = [0, 1]$.

$$P[X = 1] = p \quad (1)$$

$$P[X = 0] = 1 - p \quad (2)$$

where $0 < p < 1$. Alternative sample spaces can also be used. Consider for example, $\Omega = [\text{machine working}, \text{machine not working}]$. Needless to say that all binary sample space are equivalent and thus without loss of generality, we shall retain the traditional $\Omega = [0, 1]$. Typically the outcome $X = 1$ is associated with a success and $X = 0$ with a failure.

Binomial Random Variable

A binomial random variable Y is defined as the number of k successes in n independent realizations of identical Bernoulli random variables X . Alternatively it can also be defined as

$$Y = \sum_{i=1}^n X_i \quad (3)$$

The probability of the event $Y = k$ is given by

$$P[Y = k] = \frac{n!}{k!(n-k)!} (1-p)^{n-k} p^k \quad (4)$$

Poisson Random Variable

If we imagine each Bernoulli random variable X occurring within an interval of time Δt such that $n\Delta t = T$, and $p = \lambda\Delta t$, then

$$P[Y = k] = \frac{n!}{k!(n-k)!} (1 - \lambda\Delta t)^{n-k} (\lambda\Delta t)^k \quad (5)$$

$$P[Y = k] = \frac{n!}{k!(n-k)!} \left(\frac{n - \lambda T}{n}\right)^{n-k} \left(\frac{\lambda T}{n}\right)^k \quad (6)$$

If we take the limit as $n \rightarrow \infty$ and $\Delta t \rightarrow 0$ while keeping T fixed, we find

$$= \lim_{n \rightarrow \infty} \frac{n!}{k!(n-k)!} \left(\frac{n-\lambda T}{n}\right)^{n-k} \left(\frac{\lambda T}{n}\right)^k \quad (7)$$

$$= \frac{(\lambda T)^k e^{-\lambda T}}{k!} \lim_{n \rightarrow \infty} \frac{n!}{(n-\lambda T)^k} \quad (8)$$

$$P[Y = k] = \frac{(\lambda T)^k e^{-\lambda T}}{k!} \quad (9)$$

This describes the Poisson random variable where λ describes the probability of success ($X = 1$) per unit of time.

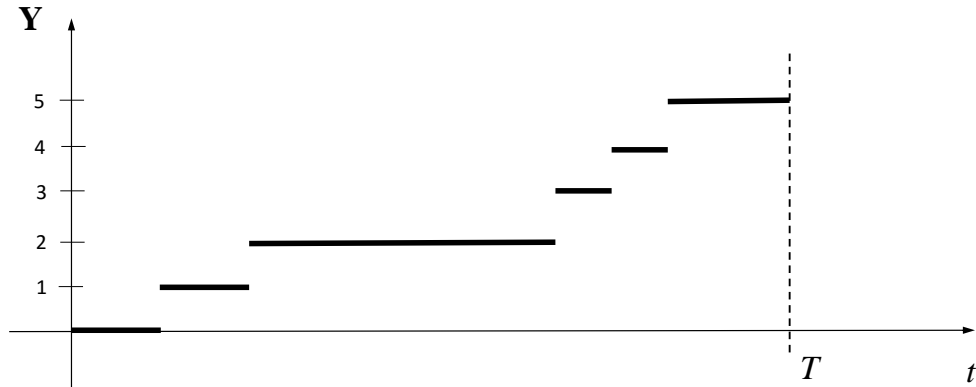


Figure 1: Sample realization of a Poisson random process